This note gives an example to help us understand why the second derivative determines local min/max and saddle points and to clear up the notes from lecture. Be sure to compare what you have written for the second derivative test in your notes agrees with the one at the bottom of this page. One issue to keep in mind is that if a first or second derivative is not continuous in a disc around \((a, b)\), then the second derivative test can not be used and you must find other techniques to detect local extrema.

Consider the function and it’s graph below:

\[
f(x, y) = 4xy - x^4 - y^4
\]

Calculating the first order partial derivatives of this function we see:

\[
\frac{\partial f}{\partial x} = 4y - 4x^3, \quad \frac{\partial f}{\partial y} = 4x - 4y^3
\]

and the gradient function is defined by:

\[
\nabla f(x, y) = <4y - 4x^3, 4x - 4y^3>
\]

Solving the equation: \(\nabla f = 0\), we see that there are flat tangent planes at \((0, 0), (1, 1),\) and \((-1, -1)\). From the graph it appears that the first point yields a saddle point while the latter two are both local maximums. To check that this interpretation is correct, we need the second order partial derivatives.

\[
\frac{\partial^2 f}{\partial x^2} = -12x^2, \quad \frac{\partial^2 f}{\partial y^2} = -12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4
\]

Lastly, at each point we must evaluate \(f_{xx}, f_{xx}f_{yy} - f_{xy}^2\) to determine their sign. Using the second derivative test below, we see that our intuition was correct.

**Definition 1 (Second Derivative Test)** Suppose \(f(x, y)\), it’s first, and it’s second partial derivatives are all continuous on a disc around \((a, b)\). Also assume that \(f_x(a, b) = f_y(a, b) = 0\). Then:

1. \(f\) has a **local minimum** at \((a, b)\) if \(f_{xx} > 0\) and \(f_{xx}f_{yy} - f_{xy}^2 > 0\) at \((a, b)\)
2. \(f\) has a **local maximum** at \((a, b)\) if \(f_{xx} < 0\) and \(f_{xx}f_{yy} - f_{xy}^2 > 0\) at \((a, b)\)
3. \(f\) has a **saddle point** at \((a, b)\) if \(f_{xx}f_{yy} - f_{xy}^2 < 0\) at \((a, b)\)
4. The test is **inconclusive** if \(f_{xx}f_{yy} - f_{xy}^2 = 0\) at \((a, b)\).