MA 125 - Calculus I
Exam 1
18 February 2010

Instructions:

• You may not use any outside assistance on this exam. You may not use books, notebooks, other people’s exams, cell phones, mp3 players, or any other materials to cheat on this exam.

• You may not use a graphing calculator on this exam.

• If you are caught cheating on any question of the exam, your grade for the entire exam will be a 0.

• You must write clearly, give exact answers and fully reduce fractions to receive full credit. Unreadable, approximate and unreduced answers will receive only partial credit.

• You must show all your work to receive full credit unless otherwise stated.

Name: KEY
Score: 102 /100 Points
1. (5 points each) Evaluate the following limits. If a limit does not exist, explain why not.

(a) \( \lim_{x \to 5} \frac{x^2 - 25}{(x-5)(x+5)} = \frac{0}{0} \) hole at \( x = 5 \)

\[
\lim_{x \to 5} \frac{x^2 - 25}{(x-5)(x+5)} = \lim_{x \to 5} \frac{2x}{x+5} = \frac{10}{10} = 1
\]

(b) \( \lim_{t \to 0} \frac{\sin 3t}{5t} = \frac{0}{0} \) \( \frac{\sin 3t}{5t} = \frac{\frac{3}{5} \sin 3t}{3t} \)

\[
\lim_{t \to 0} \frac{\sin 3t}{5t} = \lim_{t \to 0} \frac{\frac{3}{5} \sin 3t}{3t} = \frac{3}{5} \lim_{t \to 0} \frac{\sin 3t}{3t} = 1,
\]

(c) \( \lim_{x \to 2} \sqrt{4 - x^2} = \) does not exist, because the left and right handed limits do not approach the same value. The domain of this function is \([-2, 2]\), nothing to evaluate on the right side of 2.

(d) \( \lim_{h \to 0} \frac{h^2 + q - h}{h} = \) does not exist.

\[
\lim_{h \to 0} \frac{h^2 + q - h}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \]
2. (5 points each) Consider the equation \( g(x) = x^2 - 2x - 3 \).

(a) What is the average rate of change of \( g(x) \) on the interval \([0, 5]\)?

\[
\frac{\Delta y}{\Delta x} = \frac{12 - (-3)}{5 - 0} = \frac{15}{5} = 3
\]

(b) What is the equation of the secant line connecting \( g(0) \) and \( g(5) \)?

\[
g(0) = 0^2 - 2(0) - 3 = -3
\]
\[
g(5) = 5^2 - 2(5) - 3 = 12
\]

\[
y = y_1 - y_0 = m(x - x_1) \quad \text{using point } (0, -3) \quad \text{and } m = 3 \quad \text{from average rate of change}
\]
\[
y = 3x - 3
\]

(c) What is the slope of the curve \( g(x) \) at the point \((3, 0)\)?

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(3 + h)^2 - 2(3 + h) - 3}{h}
\]

\[
= \lim_{h \to 0} \frac{9 + 6h + h^2 - 6 - 2h - 3}{h}
\]

\[
= \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} \frac{h(h + 4)}{h} = \lim_{h \to 0} h + 4 = 0 + 4 = 4
\]

\[
\text{Slope} = 4
\]
3. (5 points each) Consider the following the function.

\[
l(x) = \begin{cases} 
-x & x < 0 \\
1 + x & 0 \leq x < 2 \\
2 & x = 2 \\
0 & x > 2 
\end{cases}
\]

(a) Graph the function \(l(x)\) on a Cartesian coordinate system being sure to label and scale all axis.

(b) At what values \(x = c\) does the function \(l(x)\) fail to have a two sided limit? At these values \(c\), determine the left hand limit, \(\lim_{x \to c^-} l(x)\).

\[
\begin{align*}
\lim_{x \to 0^-} l(x) &= 0 \\
\lim_{x \to 2^-} l(x) &= 3
\end{align*}
\]
4. (10 points) Determine the equations of all asymptotes for the function below.

\[ c(x) = \frac{x^2 - 5x + 4}{x^2 - 1} \]

Vertical Asymptote (V.A):

\[ \frac{x^2 - 5x + 4}{x^2 - 1} = \frac{(x-4)(x+1)}{(x-1)(x+1)} \]

There is a vertical asymptote where \( x = 1 \) and \( x = -1 \).

Horizontal Asymptote (H.A):

\[ \lim_{{x \to \pm \infty}} c(x) = 1 \]

We know that if the highest degree of the numerator and denominator are equal, the horizontal asymptote is equal to their term in third degree numerator being divided by the highest degree term in the denominator.

5. (10 points) For the function below, determine the continuous extension. Be sure to justify your answers.

\[ f(x) = \frac{x(x - 4)}{x^2 - 16} \]

\[ g(x) = \frac{x(x - 4)}{x^2 - 16} \rightarrow \text{Domain} \quad \left\{ x \mid x \neq 4, 4 \right\} \]

\[ = \frac{x(x - 4)}{(x - 4)(x + 4)} \]

\[ = \frac{x}{x + 4} \]

\( V.A \) is where \( x = -4 \).

\[ f(x) = \frac{x}{x + 4} \]

\[ g(x) = \left\{ \begin{array}{l l}
\frac{y}{x + 4} & \quad x \neq 4 \\
\frac{1}{2} & \quad x = 4
\end{array} \right. \]

After finding the domain, I simplified the function \( g(x) \) to check for any asymptotes, which occurs where \( x = -4 \). With this, I know there is a hole in the graph where \( x = 4 \). I substituted this for \( f(x) = \frac{x}{x + 4} \) and found the continuous extension at \( y = \frac{1}{2} \).
6. (3 points each) Determine if each of the statements is either true or false. You must write the entire word TRUE or FALSE to receive full credit. If you write only T or F, you will receive at most one point. You do not have to show your work for this problem.

(a) \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) = 0 \)

\[ \text{False} \]

(b) The greatest integer function or floor function, \( f(x) = \lfloor x \rfloor \), is discontinuous at all integer values of \( x \).

\[ \text{True} \]

(c) The rational function \( f(x) = \frac{2x^2 + 3}{x - 4} \) has an oblique asymptote at \( y = 2x + 8 \).

\[ \text{True} \]

(d) If \( \lim_{x \to a} t(x) = -\infty \) then \( x = a \) is a vertical asymptote to the graph \( t(x) \).

\[ \text{True} \]

(e) If the product function \( h(x) = f(x) \cdot g(x) \) is continuous at \( x = c \) then \( f(x) \) and \( g(x) \) must both be continuous at \( x = c \).

\[ \text{False} \]
7. (5 points) If \( \cos x + 3 \leq h(x) \leq 3x^2 + 4 \), determine 

\[
\lim_{x \to 0} h(x).
\]

\[
\begin{align*}
\lim_{x \to 0} \cos x + 3 &= \cos 0 + 3 = 4 & \text{because we can plug in for } \cos x \\
\lim_{x \to 0} 3x^2 + 4 &= 3(0)^2 + 4 = 4 & \text{plug in for polynomial}
\end{align*}
\]

\( \lim_{x \to 0} h(x) \) is between \( \lim_{x \to 0} \cos x + 3 \) and \( \lim_{x \to 0} 3x^2 + 4 \).

Henceforth, by way of the Sandwich Thrm, \( \lim_{x \to 0} h(x) = 0 \).

8. (5 points) Using the Intermediate Value Theorem, show that the function below has a root between \(-1\) and \(2\).

\( r(x) = x^3 - x - 1 \)

\[
\begin{align*}
\ f(-1) &= -1 + 1 - 1 = -1 \\
\ f(0) &= 0 - 0 - 1 = -1 \\
\ f(1) &= 1 - 1 - 1 = -1 \\
\ f(2) &= 8 - 2 - 1 = 5
\end{align*}
\]

\( f(-1) < 0 < f(2) \)

Therefore, a root must exist between \(-1\) and \(2\), because this is a polynomial, which is continuous on its entire domain.
9. (5 points each) Determine where the following functions are continuous.

(a) \( s(x) = \sqrt{\frac{1}{x^2}} \)

Continuous at all points \( x > 5 \). Why? What things are you e.g. p.

The square root of a negative number will produce imaginary points. At all \( x < 5 \), the output is imaginary and therefore discontinuous. At \( x = 5 \), the output is undefined.

\[ \sqrt{\frac{1}{5^2}} = \frac{1}{5} \]

\[ \sqrt{\frac{1}{x^2}} = i \sqrt{-1} = i \]

(b) \( t(\theta) = \tan \theta + \cot \theta \)

\[ t(x) = \tan x + \cot x = \frac{\sin x \cos x}{\cos x} \frac{\cos x}{\sin x} \]

\[ \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos \theta \sin \theta} \]

Nice idea. Discontinuous at \( \theta = \pi k \)

\[ \theta = \pi k - \frac{\pi}{2} \]

Where \( k \) is an integer.

Continuous for all other values \( \theta \).
10. Extra Credit: (5 points) Consider the function \( q(x) = \sqrt{x-7} \), \( x_0 = 23 \), and \( \varepsilon = 1 \). Find a number \( \delta > 0 \) such that for all \( x \):

\[
0 < |x - x_0| < \delta \Rightarrow |q(x) - L| < \varepsilon
\]

where \( L \) is the limit of \( q(x) \) when \( x \) approaches \( x_0 \).

\[
0 < |x - 23| < \delta \Rightarrow \left| \sqrt{x-7} - 4 \right| < 1
\]

\[
0 < |x - 23| < \delta \Rightarrow \left| \sqrt{x-7} - 4 \right| < 1
\]