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Recent Themes (And Theme Parks) In AP Calculus

Summary

Even though the AP Calculus Free-Response questions are not predictable, there have been some constant themes over the past few years. The Fundamental Theorem of Calculus, the area-so-far function, changing animal populations, and amusement parks have recently appeared in various contexts. This presentation will highlight some constant AP Calculus themes, discuss typical responses, and consider the grading of these theme questions.

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1. Write less, say more.

   Since powerful graphing calculators can now be used on part of the Free-Response section of the AP Calculus Exam, the emphasis has shifted *from* calculations *to* set-up. In a typical 3-4 point definite integral problem, we often awarded 2 points for the antiderivative and answer. Now, we usually award 2 points for the set up, and 1 point for the answer. This is consistent with the desire to stress and test understanding of concepts.

   Examples:

   (a) Find the volume of the solid generated ... (2002 AB1).

   $$\pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2\right)\, dx = 23.609 \qquad \text{(4 points)}$$

   (b) How many people entered the park by 5:00 pm? (2002 AB2).

   $$\int_{9}^{17} E(t)\, dt = 6004.270 \qquad \text{(3 points)}$$

   (c) Find the total distance traveled by the object (2002 AB3).

   $$\int_{0}^{4} |v(t)|\, dt = 2.387 \qquad \text{(3 points)}$$

   (d) Find the area or volume (2002B AB1).

   $$\pi \int_{0}^{A} \left((4 - 2x)^2 - \left(\frac{x^3}{1+x^2}\right)^2\right) dx = 31.884 \qquad \text{(3 points)}$$

   $$\int_{0}^{A} \left(4 - 2x - \frac{x^3}{1+x^2}\right)^2 dx = 8.997 \qquad \text{(3 points)}$$

   (e) Find the total distance traveled by the particle from $t = 0$ to $t = 4$ (2002B AB3).

   $$\int_{0}^{4} |v(t)|\, dt = 10.542 \qquad \text{(3 points)}$$

   (f) 2001 AB1: area, volume question. Nine points in three lines.

   (g) 2000 AB1: area, volume question. Nine points, four lines (need the point of intersection).

   (h) Write, but do not solve, an equation involving an integral expression that can be used to find the value of k (1999 AB2).

   $$\pi \int_{-2}^{2} [(k - x^2)^2 - (k - 4)^2]\, dx = \frac{256\pi}{5} \qquad \text{(4 points)}$$

2. Interpretation.

There has been an increased emphasis on interpretation of results. Students have been asked to explain the exact meaning of numerical answers, for example, the derivative of a function at a specific point.

Examples:

(a) Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park (2002 AB2).

There were 3725 people in the park at $t = 17$.

The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.

(b) Is the amount of pollutant increasing at time $t = 9$? Why or why not? (2002B AB2).

$P'(9) = 1 - 3e^{-0.6} = -0.646 < 0$

The amount is not increasing at this time. (1 point)

(c) Using appropriate units, explain the meaning of your answer in terms of water temperature (2001 AB2).

$P'(12) = -0.549$ °C/day

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

(d) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not? (2001 AB3)

Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

3. The area-so-far function.

There have been several problems recently that involve the area-so-far function, for example,

$$g(x) = \int_a^x f(t)\,dt$$

Students have been asked where the function g is increasing, decreasing, concave up, concave down, and location of extrema.

Examples:

(a) $g(x) = \int_0^x f(t)\,dt$ (2002 AB4)

Evaluate: $g(-1)$, $g'(-1)$, $g''(-1)$.

Increasing, concave up, sketch a graph of $y = g(x)$.

(b) $g(x) = 5 + \int_6^x f(t)\,dt$ (2002B AB4)

Evaluate: $g(6)$, $g'(6)$, $g''(6)$.

Decreasing, concave down, trapezoidal approximation.

(c) The figure above shows the graph of f', the derivative of the function f, for $-7 \leq x \leq 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

Relative maximum, concave down, absolute maximum.

(d) $g(x) = \int_1^x f(t)\,dt$ (1999 AB5)

Evaluate: $g(4)$, $g(-2)$, $g'(1)$.

Absolute minimum value of g. Points of inflection.

4. Area-volume, some with a twist.

The standard area, volume (using disks, or shells on earlier exams), and generalized volume problems are still here. These questions are usually calculator active and require a solid understanding of the concepts (and formulas). Some questions have been modified and require a clever solution. In addition, the awarding of points has become weighted toward set-up.

Examples:

(a) Area, volume of a solid of revolution. (2002 AB1)

Twist: Let h be the function given by $h(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $1/2 \leq x \leq 1$, and find the absolute maximum value of $h(x)$ on the closed interval $1/2 \leq x \leq 1$.

(b) Area, volume of a solid of revolution, generalized volume. (2002B AB1)

(c) Area (2 regions), volume of a solid of revolution. (2001 AB1)

(d) Area, volume of a solid of revolution, generalized volume. (2000 AB1)

(e) Area, volume of a solid of revolution. (1999 AB2)

Twist: There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

$$\pi \int_{-2}^{2} [(k - x^2)^2 - (k - 4)^2] \, dx = \frac{256\pi}{5}$$

(f) Area, and volume of a solid of revolution. (1998 AB1)

Twist: The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x-axis, they generate solids with equal volumes. Find the value of k.

$$\pi \int_{0}^{k} (\sqrt{x})^2 \, dx = 4\pi$$

5. Tables.

A function may be defined in several ways: analytically, graphically, using a table, or even in words. Problems containing a table of function values (or values of the derivative) at selected points are becoming routine. Students have been asked a variety of questions that require the table and other stated properties.

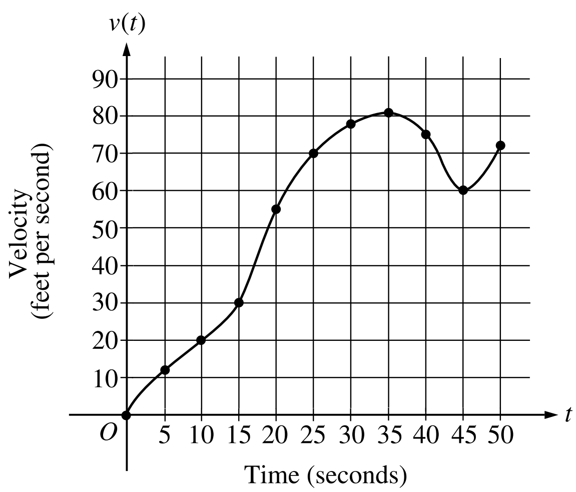
Examples:

(a) Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \leq x \leq 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \leq x \leq 1.5$. (2002 AB6)

(b) The temperature, in degrees Celsius (°C), of water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period. (2001 AB2)

(c) The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period. (1999 AB3)

(d) The graph of the velocity $v(t)$, in ft/sec, or a car traveling on a straight road, for $0 \leq t \leq 50$, is shown above, A table of values for $v(t)$, at 5 second intervals of time t, is shown to the right of the graph. (1998 AB3)



t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72

6. The integral as an accumulation function.

The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. The statement of the theorem includes the definition of an *accumulation* function. Recent AP Free Response questions have asked students to use all kinds of accumulation functions.

Examples:

(a) How many people have entered the park by 5:00 pm? (2002 AB2)

$$\int_9^{17} E(t)\,dt = 6004.270$$

Part (b): How many dollars are collected from admissions to the park on the given day?

$$15\int_9^{17} E(t)\,dt + 11\int_{17}^{23} E(t)\,dt = 104048.165$$

(b) Various particle motion problems.

What is the total distance traveled over an interval?

What is the position of the object at time $t = t_0$?

See 2002 AB3, 2002B AB3, 1999 AB1.

(c) The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant. (2002 AB2)

Is the lake safe when the number of gallons of pollutant is at its minimum?

$$P(30.174) = 50 + \int_0^{30.174} \left(1 - 3e^{-0.2\sqrt{t}}\right) dt = 35.104$$

(d) Accumulation using a graph: 2001 AB3, 2000 AB2(d).

(e) Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water. (2000 AB4)

How many gallons leak out of the tank from time $t = 0$ to $t = 3$ minutes?

Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

7. Solve a separable differential equation.

These problems appear fairly regularly. The assignment of points has also become fairly standard. Sometimes there is an extra detail to worry about, for example, use the negative square root.

Grading Standard:

1: separates variables
1: antiderivative of dy term
1: antiderivative of dx term
1: constant of integration
1: uses initial condition
1: solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration.

Note: 0/6 if no separation of variables.

Examples:

(a) $\dfrac{dy}{dx} = \dfrac{3-x}{y}$. (2002B AB5)

Note the easy way to justify the answer in part (a).

Note the need for the negative square root. Why?

(b) $\dfrac{dy}{dx} = y^2(6-2x)$. (2001 AB6)

Implicit differentiation in part (a).

(c) $\dfrac{dy}{dx} = \dfrac{3x^2}{e^{2y}}$. (2000 AB6)

Domain and range in part (b).

(d) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ (1998 AB4)

Five points for the solution to the differential equation.

Need the positive square root in this problem.

8. The Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus (parts 1 and 2) is the most important idea in the course. It connects differential calculus and integral calculus. It is very reasonable to assume the Free-Response section of the AP test will contain a question involving the FTC, in some form. Recent problems require solutions that use the FTC in a clever way.

Examples:

(a) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$.

Find the value of $H'(17)$.
At what time t, for $9 \le t \le 23$, does the model predict that the number in the park is a maximum? (2002 AB2)

(b) Evaluate $\int_0^{1.5} (3f'(x) + r)\,dx$ (2002 AB6)

(c) $g(x) = 5 + \int_6^x f(t)\,dx$ Find $g'(6)$ (2002B AB4)

9. Particle Motion.

These problems used to be very common. They still appear regularly on the AP Exam, but the functions that describe velocity, etc. have become more complicated (students can use a calculator).

Examples:

(a) $v(t) = e^{2\sin t} - 1$ (2002 AB3)

Sketch a graph of $v(t)$.

(b) $v(t) = t\sin(t^2)$ (1999 AB1)

Acceleration, position, total distance traveled.

(c) Particle motion in the plane: $x(t) = \sin(3t)$, $y(t) = 2t$. (2002B BC1)

Sketch, speed, total distance traveled.

(d) Particle motion in the plane: $\frac{dx}{dt} = \cos(t^3)$, $\frac{dy}{dt} = 3\sin(t^2)$

Tangent line, speed, total distance traveled, position.

(e) Particle motion in the plane: Velocity vector given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$ (2000 BC4)

Acceleration vector, position, tangent.

(f) See also: 1999 BC1,1998 BC6.

10. Other common themes.

    (a) Slope fields: sketch and/or interpret.

    (b) Euler's method: estimate a value after 2 steps.

    (c) Related rates: watch for negative rate of change.

    (d) Series question: convergence or divergence at the endpoints.

    (e) Common student errors.

AB-2 (2003)

The rate at which gnus are entering a combination amusement and water park on a given day is modeled by the function E defined by

$$E(t) = \frac{3230 \sin(t^2)}{1 + (9/5)e^x}.$$

The rate at which the gnus leave the same park on the same day is modeled by the function L defined by

$$L(t) = i + \frac{d * o}{n^t} - w^{a^{n^{n^a}}} + (l)(e^a)(v_e).$$

Both $E(t)$ and $L(t)$ are measured in gnus per hour and time t is measured in hours before sundown, $-16 \leq t \leq 0$. At time $t = -16$, there are no gnus in the park.

A baby gnu buys a cup of water ice in a container shaped in the form of an inverted right circular cone. Alas, the cup has a small leak. The water ice is dripping out of the cup through the hole in the bottom according to the function defined by

$$d(t) = \ln t + \sin t - (oh)(my) \cosh t.$$

The entire gnu family decides to ride the roller coaster.

(a) How many gnus are in the park at any given time. Find an expression in terms of t.

(b) What is the acceleration and velocity of the roller coaster when there is half a cup of water ice remaining?

(c) The Table on Page xx gives the centripetal acceleration of a rider on the Boom Ride at various times. Use the Trapezoidal method to approximate the amount of pollutant in the air next to the diesel motor running the ferris wheel. Use correct units.

(d) The expected profit form each gnu entering the park is \$5.75 with standard deviation \$.37. When should the park manager shut the park down and send all the gnus home in order to maximize profit? Justify your answer. What is the maximum profit? Indicate your answer to the nearest cent.

(e) Use the Intermediate Value Theorem to show there is some time during the day when the line waiting for the Swing Ride is tangent to the ice-cream stand.