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Fifty Famous Curves - Solutions

Useful Formulas

1. $\sin^2 x = \dfrac{1-\cos 2x}{2} \qquad \cos^2 x = \dfrac{1+\cos 2x}{2}$

2. The area of a region A bounded above by the graph of $y = f(x)$, below by the x-axis, and by the lines $x = a$ and $x = b$, where f is a continuous function for all $x \in [a, b]$ is given by

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx.$$

If the curve is given by parametric equations $x = f(t)$, $y = g(t)$ and is traversed once as t increases from α to β, then

$$A = \int_a^b y\;dx = \int_\alpha^\beta g(t)f'(t)\,dt,$$

or

$$A = \int_\beta^\alpha g(t)f'(t)\,dt \quad \text{if } (f(\beta), g(\beta)) \text{ is the leftmost endpoint.}$$

If A is a region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and $0 < b - a \leq 2\pi$, then

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta = \int_a^b \frac{1}{2}r^2\,d\theta.$$

3. If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If the curve is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$, then the curve is traversed once from left to right as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$, and

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

4. If a curve is described by the parametric equations $x = f(t)$ and $y = g(t)$, then the slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if} \quad \frac{dx}{dt} \neq 0.$$

The curve has a horizontal tangent when $dy/dt = 0$ (provided $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided $dy/dt \neq 0$).

Also note

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

To find a tangent line to a polar curve $r = f(\theta)$, write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Using the Product Rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

1. **Astroid**

   (a) Graph: Parametric, $[0, 2\pi]$, $[-2, 2] \times [-2, 2]$, `ZoomSqr`.

   (b) Length

$$y^{2/3} = a^{2/3} - x^{2/3} \qquad y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$$

$$y' = \frac{3}{2}\left(a^{2/3} - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(a^{2/3} - x^{2/3}\right)^{1/2}}{x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{a^{2/3} - x^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}$$

$$L = 4\int_0^a \sqrt{a^{2/3}\, x^{-2/3}}\; dx = \lim_{t\to 0^+} 4a^{1/3}\int_t^a x^{-1/3}\, dx$$

$$= 4a^{1/3} \lim_{t\to 0^+} \frac{3}{2}x^{2/3}\Big]_t^a$$

$$= 4a^{1/3} \lim_{t\to 0^+} \frac{3}{2}\left(a^{2/3} - t^{2/3}\right) = 6a$$

(c) Area

$$f(t) = a\cos^3 t \implies f'(t) = -3a\cos^2 t \sin t \qquad g(t) = a\sin^3 t$$

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

$$= 4\int_{\pi/2}^0 a\sin^3 t(-3a\cos^2 t\sin t)\,dt = 4\int_{\pi/2}^0 -3a^2\sin^4 t\cos^2 t\,dt$$

$$= -12a^2\left[\frac{t}{16} - \frac{1}{64}\sin 2t - \frac{1}{64}\sin 4t + \frac{1}{192}\sin 6t\right]_{\pi/2}^0$$

$$= -12a^2\left(-\frac{\pi}{32}\right) = \frac{12\pi a^2}{32} = \frac{3\pi a^2}{8}$$

(d) Tangent line

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

$$t = t_0 \qquad x = a\cos^3 t_0 \qquad y = a\sin^3 t_0$$

$$y - a\sin^3 t_0 = -\tan t_0(x - a\cos^3 t_0)$$

$$x\tan t_0 + y = a\sin^3 t_0 + a\sin t_0\cos^2 t_0$$

$$\begin{aligned} x\sin t_0 + y\cos t_0 &= a\sin^3 t_0\cos t_0 + a\sin t_0\cos^3 t_0 \\ &= a\sin t_0\cos t_0(\sin^2 t_0 + \cos^2 t_0) \\ &= \frac{a}{2}\sin 2t_0 \end{aligned}$$

(e) Constant distance

$$x \sin t_0 + y \cos t_0 = \frac{a}{2} \sin 2t_0$$

$$x = 0 : \quad y = \frac{a \sin 2t_0}{2 \cos t_0} = \frac{a 2 \sin t_0 \cos t_0}{2 \cos t_0} = a \sin t_0$$

$$y = 0 : \quad x = \frac{a 2 \sin t_0 \cos t_0}{2 \sin t_0} = a \cos t_0$$

Two points: $(0, a \sin t_0), \quad (a \cos t_0, 0)$

$$D = \sqrt{a^2 \cos^2 t_0 + a^2 \sin^2 t_0} = a$$

2. **Bicorn**

   (a) Graph (TI-89)

   Solve for y ($a = 1$), copy, paste, graph.

   Implicit plot.

   (b) Area

   Numerical integration

3. **Cardioid**

   (a) Length of any chord through the cusp point $= 4a$.

   (b) Area enclosed by the cardioid $= 6\pi a^2$.

   (c) Length of the curve $= 16a$.

4. **Cartesian Oval**

   (a) Graph shown: $m = 2.5,\ a = 2.5,\ c = 1.5$ try implicit plot.

   (b) Area: How?

   (c) Tangent line: implicit differentiation.

5. **Cassinian Ovals**

   (a) Area enclosed by the ovals: four curves.

   (b) Slope of the tangent line: implicit differentiation.

6. **Catenary**

(a) Length of the curve between $x = c$ and $x = d$.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Let $a = 1 \implies y = \cosh x \implies y' = \sinh x$

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \sinh^2 x}\; dx = \int_c^d \cosh x \, dx \\ &= \sinh x \Big]_c^d = \sinh d - \sinh c \end{aligned}$$

(b) Area

$$\begin{aligned} A &= \int_c^d (\cosh x - 0)\, dx \\ &= \sinh x \Big]_c^d = \sinh d - \sinh c \end{aligned}$$

(c) Equation of the tangent line.

$$y = a\cosh(x/a) \qquad y' = a\sinh(x/a) \cdot \frac{1}{a} = \sinh(x/a)$$

$m = y'(c) = \sinh(c/a)$ Point: $(c, a\cosh(c/a))$

Tangent line: $y - a\cosh(c/a) = \sinh(c/a)(x - c)$

(d) Volume

Solve: $\cosh x = 2 \implies x = \ln(2+\sqrt{3}),\ \ln(2-\sqrt{3})$ (How?)

$$\begin{aligned} V &= 2\pi \int_0^{\ln(2+\sqrt{3})} \left[(2-(-1))^2 - (\cosh x - (-1))^2\right] dx \\ &= 2\pi \int_0^{\ln(2+\sqrt{3})} (8 - \cosh^2 x - 2\cosh x)\, dx \\ &= \ldots = 3\pi(5\ln(2+\sqrt{3}) - 2\sqrt{3}) \end{aligned}$$

7. **Cayley's Sextic** (Let $a = 1$)

(a) Graph

Polar form: r: $[0, 3\pi]$, $[-1.5, 4.5] \times [-3.5, 3.5]$.

(b) Length of the curve

$r = 4\cos^3(\theta/3) \qquad r^2 = 16\cos^6(\theta/3)$

$$\frac{dr}{d\theta} = 12\cos^2(\theta/3) \cdot (-\sin(\theta/3)) \cdot \frac{1}{3} = -4\sin(\theta/3)\cos^2(\theta/3)$$

$$\left(\frac{dr}{d\theta}\right)^2 = 16\sin^2(\theta/3)\cos^4(\theta/3)$$

$$\begin{aligned}
L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= 2\int_0^{3\pi/2} \sqrt{16\cos^6(\theta/3) + 16\sin^2(\theta/3)\cos^4(\theta/3)}\, d\theta \\
&= 2\int_0^{3\pi/2} \sqrt{16\cos^4(\theta/3)\left[\cos^2(\theta/3) + \sin^2(\theta/3)\right]}\, d\theta \\
&= 2\int_0^{3\pi/2} 4\cos^2(\theta/3)\, d\theta = 8\int_0^{3\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta/3}{2}\right) d\theta \\
&= 4\left[\theta + \frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{3\pi/2} \\
&= 4\left[\left(\frac{3\pi}{2} + \frac{3\pi}{2}\sin(\pi)\right) - (0 + 0)\right] \\
&= 4\left(\frac{3\pi}{2}\right) = 6\pi
\end{aligned}$$

8. **Cissoid of Diocles**

Let $a = 1$ $\quad y^2 = \dfrac{x^3}{2-x}$

(a) Domain questions. Asymptotic behavior.

(b) Area bounded by the curve and its asymptote.

$$
\begin{aligned}
A_{\mathrm{H}} &= \int_0^2 \left(\frac{x^3}{2-x}\right)^{1/2} dx \\
&= \lim_{b\to 2^-} \int_0^b \left(\frac{x^3}{2-x}\right)^{1/2} dx \\
&= \ldots = 4.71239 \qquad \left(= \frac{3\pi}{2}\right)
\end{aligned}
$$

Error?

$$r = 2\tan\theta\sin\theta = \frac{2\sin^2\theta}{\cos\theta}$$

$$A_{\mathrm{H}} = \frac{1}{2}\int_0^{\pi/2} 4\frac{\sin^4\theta}{\cos^2\theta}\, d\theta = \ldots = \infty$$

(c) A closer look at the definite integral.

$$A_{\mathrm{H}} = \lim_{b\to 2^-} \int_0^b \left(\frac{x^3}{2-x}\right)^{1/2} dx \qquad x = 2\sin^2\theta$$

$$dx = 4\sin\theta\cos\theta$$

$$= \lim_{b\to\pi/2} \int_0^b \frac{2\sqrt{2}\sin^3\theta}{\sqrt{2}\cos\theta} \cdot 4\sin\theta\cos\theta\, d\theta \qquad x = 0 \implies \theta = 0$$

$$x = 2 \implies \theta = \pi/2$$

$$= 8 \lim_{b\to\pi/2} \int_0^b \sin^4\theta\, d\theta$$

$$= 8 \lim_{b\to\pi/2} \left[-\frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^b$$

$$= 8\left(\frac{3\pi}{16}\right) = \frac{3\pi}{2}$$

9. **Cochleoid**

(a) Sketch:

Polar form: $r = \dfrac{\sin\theta}{\theta} \quad [-12\pi, 12\pi]$

(b) Area:

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi} \frac{\sin^2\theta}{\theta^2}\, d\theta = \lim_{a\to 0}\int_a^{\pi} \frac{\sin^2\theta}{\theta^2}\, d\theta = \,?$$

(c) Tangent line:

$$r = a\frac{\sin\theta}{\theta} \qquad \frac{dr}{d\theta} = a\left[\frac{\theta\cos\theta - \sin\theta}{\theta^2}\right]$$

$$\frac{dy}{dx} = \frac{a\left[\dfrac{\theta\cos\theta - \sin\theta}{\theta^2}\right]\sin\theta + a\dfrac{\sin\theta}{\theta}\cdot\cos\theta}{a\left[\dfrac{\theta\cos\theta - \sin\theta}{\theta^2}\right]\cos\theta - a\dfrac{\sin\theta}{\theta}\cdot\sin\theta}$$

Let $\theta = \pi/2 \implies r = \dfrac{2a}{\pi}$

$$\frac{dy}{dx} = \frac{\dfrac{-a}{(\pi/2)^2}}{-\dfrac{2a}{\pi}} = \frac{2}{\pi}$$

10. **Conchoid**

(a) Sketch:

Use $x = a + \cos t, \quad y = a \tan t + \sin t$

$a = -2, -1, -0.5, -0.2, 0, 0.5, 1, 2, 3$

$t: \ [0, 2\pi], \qquad \text{tstep} = \pi/20 \qquad$ Connected versus dot mode.

(b) Area:

$$\begin{aligned} A &= \int_{\pi/2}^{0} y\, dx \\ &= \lim_{a \to \pi/2^-} \int_a^0 (-2\tan t + \sin t)(-\sin t)\, dt \\ &= \lim_{a \to \pi/2^-} \left[-\frac{t}{2} + 4\tanh^{-1}\left(\tan\left(\frac{t}{2}\right)\right) - 2\sin t + \frac{1}{4}\sin(2t) \right]_a^0 = -\infty \end{aligned}$$

11. **Conchoid of de Sluze**

(a) Graph:

$$a = -1, \quad k = -2 \qquad y = \pm \frac{x\sqrt{1-x}}{\sqrt{x+1}}$$

(b) Area of the loop

$$A = 2\int_0^1 \frac{x\sqrt{1-x}}{\sqrt{x+1}}\,dx$$

$$= 8\int_1^0 \frac{-u^2(1-u^2)}{(1+u^2)^3}\,du \qquad u = \sqrt{\frac{1-x}{1+x}}$$

$$= 2\left[-\frac{u(1+3u^2)}{(1+u^2)^2} + \tan^{-1} u\right]_1^0$$

$$= 2\left(1 - \frac{\pi}{4}\right)$$

(c) Area between the curve and the asymptote

$$A = \int_{-1}^0 \frac{-x\sqrt{1-x}}{\sqrt{1+x}}\,dx$$

$$= \lim_{a \to -1^+} \int_a^0 -x\sqrt{\frac{1-x}{1+x}}\,dx \qquad u = \sqrt{\frac{1-x}{1+x}}$$

$$= 1 + \frac{\pi}{4}$$

12. **Devil's Curve**

Sketch

Let $a = -1$, $b = 1.5$: $\quad y^4 - x^4 - y^2 + 1.5x^2 = 0$

Solve for y:

$$y = -0.5\sqrt{2 - 2\sqrt{1 - 4\,x^2\,\left(1.5 - x^2\right)}}$$

$$y = 0.5\sqrt{2 - 2\sqrt{1 - 4\,x^2\,\left(1.5 - x^2\right)}}$$

$$y = -0.5\sqrt{2 + 2\sqrt{1 - 4\,x^2\,\left(1.5 - x^2\right)}}$$

$$y = 0.5\sqrt{2 + 2\sqrt{1 - 4\,x^2\,\left(1.5 - x^2\right)}}$$

13. **Double Folium**

(a) Area of a loop ($a = 1$)

$$A = \frac{1}{2}\int_0^{\pi/2} 16\cos^2\theta\sin^4\theta\,d\theta$$

$$= 8\left[\frac{1}{192}(12\theta - 3\sin(2\theta) - 3\sin(4\theta) + \sin(6\theta)\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

(b) Points on the curve where the tangent line is vertical

$$\frac{dr}{d\theta} = 4a(-\sin^3\theta + \cos\theta\cdot 2\sin\theta\cos\theta)$$

$$= 4a(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

$$\frac{dx}{d\theta} = 4a(-\sin^3\theta + 2\sin\theta\cos^2\theta)\cos\theta - 4a\cos\theta\sin^2\theta\sin\theta$$

$$= -4a\sin^3\theta\cos\theta + 8a\sin\theta\cos^3\theta - 4a\sin^3\theta\cos\theta$$

$$= 8a(-\sin^3\theta\cos\theta + \sin\theta\cos^3\theta)$$

$$= 8a\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$

$$= 8a\sin\theta\cos\theta\cos 2\theta$$

Candidates: $0, \quad \pi, \quad \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{\pi}{4}, \quad \frac{3\pi}{4}$

14. **Durer's Shell Curves**

(a) Implicit plot: $a = 3,\ b = 13$

(b) Slope of the tangent line to the curve at any point.

$$x = 5, \quad y = \frac{225}{17} \quad \left(\text{or} \quad \frac{-33}{7}\right)$$

$$m = -\frac{504701}{69972}$$

15. **Figure Eight**

(a) Graph: $a = 4, \quad y = \pm \dfrac{x\sqrt{16 - x^2}}{4}$

(b) Area of a loop

$$A = 4\int_0^a \frac{x\sqrt{a^2 - x^2}}{a}\ dx = \frac{4}{a}\left(-\frac{1}{2}\right)\frac{2}{3}(a^2 - x^2)^{3/2}\Big]_0^a$$

$$= \frac{-4}{3a}(0 - (a^2)) = \frac{-4}{3a}(-a^3) = \frac{4}{3}a^2$$

(c) Volume

$$V = \pi\int_0^a \left(\frac{x\sqrt{a^2 - x^2}}{a}\right)^2\ dx = \frac{\pi}{a^2}\int_0^a x^2(a^2 - x^2)\ dx$$

$$= \frac{\pi}{a^2}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{2\pi a^3}{15}$$

16. **Epicycloid**

   (a) Graph: $a = 8,\ b = 5 \quad t : [0, 10\pi]$

   (b) Length and area of the curve

   If $a = (m-1)b \qquad L = 8mb \qquad A = \pi b^2(m^2 + m)$

17. **Epitrochoid**

   Graph: $a = 5,\ \ b = 3,\ \ c = 5 \qquad t : [0, 6\pi]$

18. **Equiangular Spiral**

   (a) Graph: $a = 1,\ \ b = 7\pi/16 \qquad (b = \pi/2$ - circle)

   (b) Near the origin - infinite spiral. Why?

   (c) Length of curve from P to origin.

$$r = d = ae^{\theta \cot b} \quad \Longrightarrow \quad \frac{d}{a} = e^{\theta \cot b}$$

$$\ln d - \ln a = \theta \cot b \quad \Longrightarrow \quad \theta = \frac{\ln d - \ln a}{\cot b}$$

$$r^2 = a^2 e^{2\theta \cot b} \qquad \frac{dr}{d\theta} = a(\cot b)e^{\theta \cot b}$$

$$
\begin{aligned}
L &= \int_{-\infty}^{\theta} \sqrt{a^2\, e^{2\theta \cot b} + a^2 \cot^2 b\, e^{2\theta \cot b}}\; d\theta \\
&= \lim_{t \to -\infty} \int_{t}^{\theta} a\, e^{\theta \cot b} \cdot \csc b\, d\theta \\
&= \lim_{t \to -\infty} \left[a\, \csc b \cdot \tan b \cdot e^{\theta \cot b} \right]_0^{\theta} \\
&= a\, \sec b \lim_{t \to -\infty} \left[e^{\theta \cot b} - e^{t \cot b} \right] \\
&= a\, \sec b \cdot \frac{d}{a} = d\, \sec b
\end{aligned}
$$

19. **Fermat's Spiral**

    Graph: $a = 1 \quad r = \pm\sqrt{\theta}$

20. **Folium**

    (a) Graph: $a = 1,\ b = 4,$ implicit plot

    (b) Area of the loop

$$
\begin{aligned}
A &= 2\frac{1}{2} \int_{\pi/2}^{\pi} (-b\cos\theta + 4a\cos\theta\sin^2\theta)^2\; d\theta \\
&= \left[a^2\, t - a\, b\, t + \frac{b^2\, t}{2} - \frac{a^2\, \cos(t)\, \sin(t)}{2} + \frac{b^2\, \sin(2\, t)}{4} - \frac{a^2\, \sin(4\, t)}{4} \right. \\
&\quad \left. + \frac{a\, b\, \sin(4\, t)}{4} + \frac{a^2\, \sin(6\, t)}{12} \right]_{\pi/2}^{\pi} \\
&= \frac{\pi}{4}(2a^2 - 2ab + b^2)
\end{aligned}
$$

21. **Folium of Descartes**

(a) Graph: $a = 1,\ \ t : [-10, 10]$

(b) Equation of the asymptote

$$\lim_{t\to -1} \frac{dy/dt}{dx/dt} = \lim_{t\to -1} \frac{\dfrac{-3\,\left(-2\,t+t^4\right)}{\left(1+t^3\right)^2}}{\dfrac{-3\,\left(-1+2\,t^3\right)}{\left(1+t^3\right)^2}}$$

$$= \lim_{t\to -1} \frac{-2t+t^4}{-1+2t^3} = \frac{3}{-3} = -1$$

$$\lim_{t\to -1}(x+y) = \lim_{t\to -1} \frac{3a(t^2+t)}{t^3+1} = \lim_{t\to -1} \frac{3a(2t+1)}{3t^2} = \frac{3a(-1)}{3} = -a$$

Line: $x + y = -a$

(c) Equation of the tangent line when $t = p$

$$\frac{dy/dt}{dx/dt} = \frac{t^4-2t}{2t^3-1}$$

Line: $(2p^3-1)y - p(p^3-2)x = -3ap^2$

22. **Freeth's Nephroid**

(a) Graph: $a = 1,$ polar, $\theta : [0, 4\pi]$

(b) Length of the curve

23. **Hyperbolic Spiral**

(a) Graph: $a = 2,$ $\theta : [0, 8\pi]$ As $\theta \to \infty$, $r \to 0$

What if $\theta < 0$?

(b) Length of the curve between two points

$$r^2 = \frac{a^2}{\theta^2} \qquad \frac{dr}{d\theta} = \frac{-a}{\theta^2} \qquad \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2}{\theta^4}$$

$$L = \int_\alpha^\beta \sqrt{\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}}\, d\theta = a \int_\alpha^\beta \frac{1}{\theta^2}\sqrt{\theta^2 + 1}\, d\theta$$

$$= a \left(\frac{-\sqrt{1+\theta^2}}{\theta} + \theta \ln\left|\sqrt{\theta^2+1} + \theta\right| \right)\Bigg]_\alpha^\beta$$

24. **Hypocycloid**

Graph: $a = 5,$ $b = 3,$ $t : [0, 6\pi]$

25. **Hypotrochoid**

Graph: $a = 5,$ $b = 7,$ $c = 2.2,$ $t : [0, 14\pi]$

26. **Involute of a Circle**

Graph: $a = 1/4, \ t : [0, 6.5\pi]$

$r = \pm\sqrt{\theta^2 + 1}$

27. **Kampyle of Eudoxus**

(a) Graph: $a = 1, \ b = 1, \ \theta : [-\pi/2, 3\pi/2]$

(b) Asymptotes?

Consider $y = \sqrt{x^4 - x^2} = |x|\sqrt{x^2 - 1}$

28. **Kappa Curve**

Graph: $a = 1, \ \theta : [0, 2\pi] \quad \theta \neq 0, \pi$

29. **Lamé Curves**

(a) Implicit plot: $a = 2, \ b = 1, \ n = 4$ (Solve for y)

(b) Area

$$y = \left(b^n\left(1 - \frac{x^n}{a^n}\right)\right)^{1/n}$$

$$A = 4\int_0^a \left(b^n\left(1 - \frac{x^n}{a^n}\right)\right)^{1/n} \ dx =$$

30. **Lemniscate of Bernoulli**

(a) Graph: polar or Cartesian (solve for y, $a = 1$)

(b) Area

$$A \,=\, 4 \cdot \frac{1}{2} \int_0^{\pi/4} \cos^2(2\theta)\, d\theta = 2 \int_0^{\pi/4} \frac{1}{2}(1 + \cos(4\theta)\, d\theta$$

$$= \left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4} = \frac{\pi}{4}$$

(c) Length of the curve

$$L = 4 \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4 \int_0^{\pi/4} \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta =$$

(d) Volume (d)

$$V \,=\, \pi \int_0^1 \left(-\frac{1}{2} - x^2 + \frac{1}{2}\sqrt{1 + 8x^2}\right)\, dx = \frac{\pi}{48}\left(3\sqrt{2}\ln(2\sqrt{2} + 3) - 4\right)$$

31. **Lissajous Curves**

(a) Graph: $a = 1,\ b = 2,\ c = 1,\ n = 3,\ t : [0, 2\pi]$

(b) Length of the curve

$$\frac{dx}{dt} = a\cos(nt + c) \cdot (-\sin(nt + c))(n)$$

$$\frac{dy}{dt} = b\cos t$$

$$L = \int_0^{2\pi} \sqrt{a^2 n^2 \sin^2(nt + c)\cos^2(nt + c) + b^2\cos^2 t}\ dt$$

(Numerical approximation?)

32. **Lituus**

(a) Graph: $a = 1,\ \theta : [0, 25],$ As $\theta \to \infty,\ r \to 0.$

(b) Length of the curve

$$L = \int_\alpha^\beta \sqrt{\frac{1}{\theta} + \frac{1}{4\theta^3}}\ d\theta$$

(c) Area

$$A = \frac{1}{2}\int_0^{\pi/2} \frac{a^2}{\theta}\ d\theta = \lim_{t \to 0^+} \frac{a^2}{2}\int_t^{\pi/2} \frac{1}{\theta}\ d\theta$$

$$= \frac{a^2}{2}\lim_{t \to 0^+} \ln|\theta|\Big]_t^{\pi/2}$$

$$= \frac{a^2}{2}\lim_{t \to 0^+} [\ln(\pi/2) - \ln t] \qquad \text{(diverges)}$$

33. **Nephroid**

(a) Graph: $a = 1, \;\; t : [0, 2\pi]$

(b) Length of the curve

$$x = a(3\cos t - \cos 3t) \qquad\qquad y = a(3\sin t - \sin 3t)$$

$$x' = 3a(-\sin t + \sin 3t) \qquad\qquad y' = 3a(\cos t - \cos 3t)$$

$$L = 2\int_0^{\pi} \sqrt{(x')^2 + (y')^2}\; dt$$

$$= 6a\sqrt{2}\int_0^{\pi} \sqrt{1 - (\sin t \sin 3t + \cos t \cos 3t)}\; dt$$

$$= 6a\sqrt{2}\int_0^{\pi} \sqrt{1 - \cos 2t}\; dt = 12a\int_0^{\pi} \sqrt{\frac{1 - \cos 2t}{2}}\; dt$$

$$= 12a\int_0^{\pi} \sin t\, dt = 12a[-\cos t]_0^{\pi}$$

$$= -12a(\cos\pi - \cos 0) = -12a(-2) = 24a$$

(c) Enclosed area

$$A = 4\int_{\pi/2}^{0} a(3\sin t - \sin 3t)(3a(-\sin t + \sin 3t)\, dt$$

$$= 12a^2\int_{\pi/2}^{0} (-3\sin^2 t + 4\sin t \sin 3t - \sin^2 3t)\, dt$$

$$= \ldots = 12a^2\pi$$

34. **Newton's Diverging Parabolas**

Graph: $a = 1.2, \;\; b = 1, \;\; c = .95$ Implicit plot, solve for y?

35. **Pearls of Sluze**

Graph: $a = 4, \;\; n = 4, \;\; k = 2, \;\; p = 3, \;\; m = 2$

36. **Pear-shaped Quartic**

(a) Graph: $y = \pm\dfrac{\sqrt{x^3(a-x)}}{b}$ Let $a = 2$ and $b = 1$.

(b) Area

$$A = 2\int_0^a \frac{1}{b}\sqrt{x^3(a-x)}\;dx = \frac{2}{b}\cdot\frac{1}{16}a^3\pi = \frac{a^3\pi}{8b} \qquad (a = 2, \;\; b = 1?)$$

(c) Near the origin

$$y' = \frac{1}{b}\cdot\frac{1}{2}(ax^3 - x^4)^{-1/2}(3ax^2 - 4x^3) = \frac{3ax^2 - 4x^3}{2b\sqrt{ax^3 - x^4}}$$

$$\lim_{x\to 0^+}\frac{3ax^2 - 4x^3}{2b\sqrt{ax^3 - x^4}} \qquad \text{(Calculator)}$$

37. **Plateau Curves**

Graph: $m = 5, \;\; n = 3, \;\; a = 1, \;\; t : [0, \pi]$

38. **Rhodonea Curves**

(a) Graph: $a = 1, \; k = 5, \; \theta : [0, \pi]$ Try $k = \sqrt{2}, \; \theta : [0, 20\pi]$

(b) Area of (1/2) loop

$$A = \frac{1}{2} \int_0^{\pi/2k} a^2 \sin^2(k\theta)\, d\theta = \frac{a^2}{2} \int_0^{\pi/2k} \frac{1}{2}(1 - \cos(2k\theta)\, d\theta$$

$$= \frac{a^2}{4} \left[\theta - \frac{1}{2k} \sin(2k\theta) \right]_0^{\pi/2k}$$

$$= \frac{a^2}{4} \left(\frac{\pi}{2k} \right) = \frac{\pi a^2}{8k}$$

39. **Right Strophoid**

(a) Graph: $a = 1, \; \theta : [-\pi/2, \pi/2]$

(b) Sign chart for $f(x)$

(c) Area of (1/2) loop

$$A = \int_0^a \sqrt{\frac{x^2(a - x)}{a + x}}\, dx = 2 \left(a^2 - \frac{1}{4} a^2 \pi \right)$$

$$= 2a^2 \left(1 - \frac{\pi}{4} \right)$$

40. **Serpentine**

(a) Graph: $y = \dfrac{a^2x}{x^2 + ab} \quad a = 1,\ b = 1/4$

(b) Horizontal tangent line

$$y' = \frac{-a^2(x^2 - ab)}{(x^2 + ab)^2} \qquad x = \pm\sqrt{ab} \quad (ab > 0)$$

(c) Inflection points

$$y'' = \frac{2a^2x(x^2 - 3ab)}{(x^2 + ab)} \qquad x = 0,\ \pm\sqrt{3ab}$$

(d) Area

$$A = \int_0^\infty \frac{a^2x}{x^2 + ab}\ dx = \lim_{t\to\infty} \frac{a^2}{2} \ln|x^2 + ab|\Big]_0^t = \infty$$

41. **Sinusoidal Spirals**

Graph: $a = 1,\ p = 1/16,\ \theta : [-4\pi, 4\pi]$

42. **Spiral of Archimedes**

(a) Note: $\theta = 0 \implies r = 0$ origin

(b) Graph: $a = 1/4,\ \theta : [0, 12\pi]$

(c) Distance between windings

$$D = a(\theta + 2\pi) - a\theta = 2a\pi$$

(d) Length of the curve

$$r = a\theta \qquad\qquad r' = a$$

$$r^2 = a^2\theta^2 \qquad\qquad (r')^2 = a^2$$

$$L = \int_0^\beta \sqrt{a^2\theta^2 + a^2}\, d\theta$$

$$= a\int_0^\beta \sqrt{1+\theta^2}\, d\theta \qquad\qquad (\theta = \tan t)$$

$$= \ldots = \frac{a}{2}\left[\beta\sqrt{1+\beta^2} + \ln\left(\sqrt{1+\beta^2}\right) + \beta\right]$$

(e) Area between windings

$$A = \frac{1}{2}\int_t^{t+2\pi} a^2\theta^2\, d\theta$$

$$= \frac{a^2}{2}\cdot\frac{\theta^3}{3}\Big]_t^{t+2\pi}$$

$$= \frac{a^2}{6}\left[(t+2\pi)^3 - t^3\right] = \frac{a^2}{6}\left[6\pi t^2 + 12\pi^2 t + 8\pi^3\right]$$

43. **Spiric Sections**

(a) Graph: $a = 1,\ c = 1.9,\ r = 3$ Implicit plot

(b) Solve for y

$$y = -\sqrt{a^2 - c^2 - r^2 - x^2 - 2\sqrt{c^2\,r^2 + r^2\,x^2}}$$

$$y = \sqrt{a^2 - c^2 - r^2 - x^2 - 2\sqrt{c^2\,r^2 + r^2\,x^2}}$$

$$y = -\sqrt{a^2 - c^2 - r^2 - x^2 + 2\sqrt{c^2\,r^2 + r^2\,x^2}}$$

$$y = \sqrt{a^2 - c^2 - r^2 - x^2 + 2\sqrt{c^2\,r^2 + r^2\,x^2}}$$

(c) $c = r + a \implies$ one point

$c > r + a \implies$ no solutions

44. **Talbot's Curve**

Graph: $a = 1,\ b = 1,\ f = \sqrt{9/10}, \quad t : [0, 2\pi]$

45. **Tractix**

Graph: $t : [-10, 10]$

46. **Tricuspoid**

(a) Graph: $a = 1, \; t : [0, 2\pi]$ Parametric equations

(b) Length of the curve

$x(t) = a(2\cos t + \cos 2t)$ $\qquad y(t) = a(2\sin t - \sin 2t)$

$x'(t) = a(-2\sin t - 2\sin 2t)$ $\qquad y'(t) = a(2\cos t - 2\cos 2t)$

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\; dt$$

$$= \int_0^{2\pi} \sqrt{8a^2(1 - \cos 3t)}\; dt = \int_0^{2\pi} \sqrt{8a^2 \cdot 2\sin^2\left(\frac{3t}{2}\right)}\; dt$$

$$= \int_0^{2\pi} 4a\sin(3t/2)\, dt = 4a(-\cos(3t/2) \cdot 2)\,\Big]_0^{2\pi}$$

$$= -8a(-1 - 1) = 16a$$

(c) Area enclosed by the curve

$$A/2 = \int_\pi^0 a(2\sin t - \sin 2t) \cdot a(-2\sin t - 2\sin 2t)\, dt$$

$$= \int_\pi^0 \left(-4a^2\sin^2 t - 2a^2 \sin t \sin 2t + 2a^2 \sin 2t\right)\, dt$$

$$= -a^2\, t - a^2\, \sin t + a^2\, \sin 2t + \frac{a^2\, \sin 3t}{3} - \frac{a^2\, \sin 4t}{4}\,\Big]_\pi^0$$

$$= a^2\pi$$

(d) Distance: $D = 4a$

47. **Trifolium**

(a) Graph: $a = 1,\ \theta : [0, \pi]$

(b) Area of a loop

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/6} a^2 \cos^2 \theta (4 \sin^2 \theta - 1)^2 \ d\theta = \frac{\pi a^2}{12}$$

(c) Length of the curve

$$L = 6 \int_0^{\pi/6} \sqrt{r(\theta)^2 + r'(\theta)^2} \ d\theta \qquad \text{(Calculator?)}$$

48. **Trisectrix of Maclaurin**

(a) Solve for y: $\quad y = \pm x \sqrt{\dfrac{3a - x}{a + x}}$ (Graph)

(b) Area of the loop

$$A = 2 \int_0^{3a} x \sqrt{\frac{3a - x}{a + x}} \ dx$$

$$= 2 \cdot \frac{1}{2} (a + x)(-3a + x) \sqrt{\frac{3a - x}{a + x}} \ \Bigg]_0^{3a} = 3\sqrt{3}\, a^2$$

49. **Tchirnhaus's Cubic**

Solve for y : $\quad y = \pm\dfrac{(x-a)\sqrt{x}}{\sqrt{3a}} \qquad a = 1$

50. **Witch of Agnesi**

(a) Note: Looks like a *normal* curve

(b) Graph: Solve for y, $\;a = 1$

(c) Inflection points

$$y'' = \frac{2a^3(3x^2 - a^2)}{(x^2 + a^2)^3} \implies x = \pm\frac{3a}{\sqrt{3}} \quad y = \frac{a}{4}$$

(d) Area

$$A = \int_0^\infty \frac{a^3}{x^2 + a^2}\,dx$$

$$= \lim_{t\to\infty}\left[a^3 \cdot \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_0^t$$

$$= a^2 \lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{a}\right) - \tan^{-1} 0\right]$$

$$= a^2 \cdot \frac{\pi}{2} = \frac{\pi a^2}{2}$$