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Let's Get Series(ous)

Summary

Presenting infinite series can be (used to be) a tedious and prescriptive task. A recent approach to this material allows students to discover the calculus of infinite series, to proceed intuitively, to accept what appears natural, and to dismiss other ideas that simply do not work. This eventually leads to theoretical mathematics and the important question of convergence. This presentation will include an introduction to infinite series similar to FDWK, an approach many AP Calculus teachers appreciate and enjoy.

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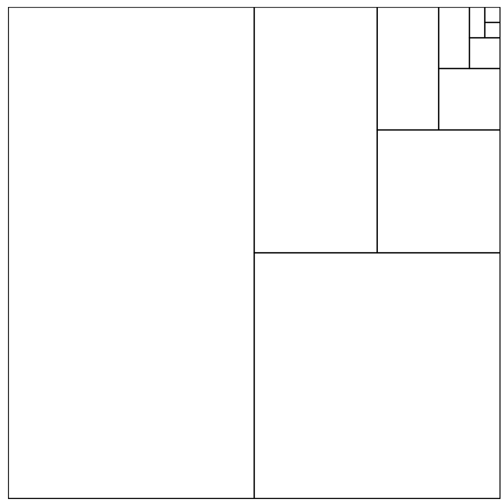
Background

Summing an infinite series can be strange.
It it not the same as summing a finite number of terms.

Consider the following examples.

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$

   Illustration:



2. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$

3. $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

   Is this 0? Or 1? Or 2? Or some other number?

Observations

1. The rules for adding a finite number of terms don't work when adding an infinite number of terms.

2. For example, the associate property doesn't seem to hold.

3. A finite sum of real numbers produces a real number.

4. An infinite sum of real numbers produces ??

Power Series

Infinite Series
An **infinite series** is an expression of the form $$a_1 + a_2 + a_3 + \cdots + a_n + \cdots, \quad \text{or} \quad \sum_{k=1}^{\infty} a_k.$$ The numbers $a_1, a_2, a_3, \ldots$ are the **terms** of the series; a_n is the ***n*th term**

The **partial sums** associated with an infinite series form a sequence of real numbers.

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} a_k \end{aligned}$$

Note:

1. Each term in the sequence of partial sums is a finite sum.
2. If the sequence of partial sums las a limit S (converges) as $n \to \infty$, then the series **converges** to S.

   If the infinite series converges, we write
   $$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = S.$$
3. Otherwise, the series **diverges**.

Example: Does the series $1 - 2 + 3 - 4 + 5 - 6 + \cdots$ converge or diverge?

Example: Does the series

$$\frac{73}{100} + \frac{73}{10000} + \frac{73}{1000000} + \cdots$$

converge or diverge? Consider the sequence of partial sums.

Remarks:

1. A geometric series is an important example of an infinite series.
2. Each term is obtained from the preceding term by multiplying by the same number r.

Definition

The **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

converges to the sum $\dfrac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

Note:

1. In words, the sum is $\dfrac{\text{first term}}{1 - \text{common ratio}}$
2. The interval $-1 < r < 1$ is the **interval of convergence**.
3. What happens if $r = 1$?
4. *Prove* this result.

Example: Determine whether each series converges or diverges. If it converges, find the sum.

1. $\displaystyle\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

2. $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \cdots$

3. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{4}{7}\right)$

4. $\dfrac{\pi}{2} + \dfrac{\pi^2}{4} + \dfrac{\pi^3}{8} + \cdots$

Example: If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$.

Remarks:

1. The expression on the right has domain $x \neq 1$.
2. The expression on the left has domain $|x| < 1$, the interval of convergence.
3. The equality holds on the intersection of the domains, where both sides are defined. On this domain, the series is indeed the function $1/(1-x)$.
4. The partial sums are polynomials. Consider some graphs.
5. This expression is like a polynomial, but its infinite!

Definition

An expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is a **power series centered at $x = 0$**. An expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is a **power series centered at $x = a$**. The term $c_n(x-a)^n$ is the **nth term**; the number a is the **center**.

Note:

1. The geometric series $\sum_{n=0}^{\infty} x^n$ is a power series centered at $x = 0$.
2. A power series either (a) converges for all x, (b) converges on a finite interval with center a, or (c) converges only at $x = a$.

Remarks:

1. The partial sums of a power series are polynomials. And, we can apply calculus techniques to polynomials.
2. It seems reasonable that (some of) these calculus techniques (like differentiation and integration) should apply to power series.

Example: Given $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ on the interval $(-1, 1)$.

Find a power series for $\ln(1-x)$. What about the interval of convergence? (Graphs)

Example: Given that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots, \quad -1 < x < 1$$

find a power series to represent $\dfrac{1}{(1+x)^2}$. What about the interval of convergence?

Theorem

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$

converges for $|x-a| < R$, then the series

$\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots,$

obtained by differentiating the series for r term by term, converges for $|x-a| < R$ and represents $f'(x)$ on the interval. If the series for f converges for all x, then so does the series for f'.

Theorem

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$

converges for $|x-a| < R$, then the series

$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + c_n\frac{(x-a)^{n+1}}{n+1} + \cdots,$

obtained by integrating the series for f term by term, converges for $|x-a| < R$, and represents $\int_a^x f(t)\,dt$ on that interval. If the series for f converges for all x, then so does the series for the integral.

Example: Use differentiation to find a series for $f(x) = \frac{2}{(1-x)^3}$.
What is the interval of convergence?

Taylor Series

Background:

1. We were able to find power series representations for certain functions.

2. Which functions have power series representations? How do we find these representations?

Example: Construct a polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ such that
$P(0) = 1,\ P'(0) = 1,\ P''(0) = 2,\ P'''(0) = 3,\ P^{(4)}(0) = 5.$

Note:

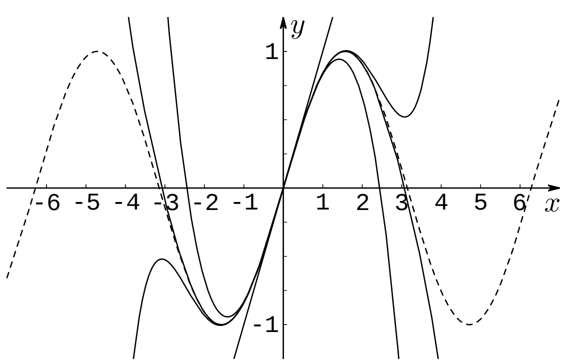
1. The coefficient of the x^n term in the polynomial is $P^{(n)}(0)/n!$.

2. We can use this fact to construct a polynomial that matches the behavior of, say, $f(x) = e^x$.

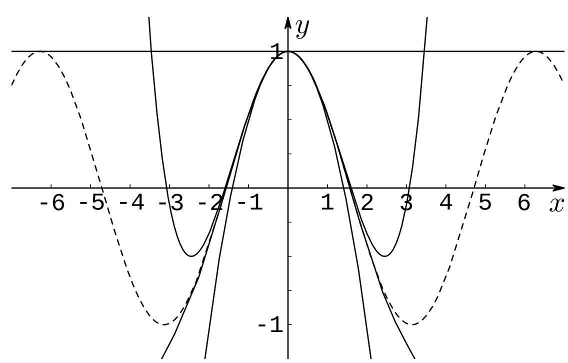
3. We are constructing an nth degree Taylor polynomial at $x = 0$.

4. If we continue to add terms, we get the Taylor series.

Example: Construct some Taylor polynomials for $\sin x$ and $\cos x$ about $x = 0$.



$f(x) = \sin x$



$f(x) = \cos x$

Note:

1. These polynomials are constructed to act like the original function near a point. The only information we really use to compute the coefficients are the derivatives at 0.

2. Remarkably, the information at $x = 0$ produces a series that looks like the sine near the origin, and appears to be more and more like the sine everywhere.

3. It seems we can construct an entire function by knowing its behavior at a single point.

4. Convergence is an infinite process. The nth order Taylor polynomial is still not perfect.

5. But, we can approximate the sine of any number to any accuracy (given enough patience).

Definition

Let f be a function with derivatives of all orders throughout some open interval containing 0. The **Taylor series generated by f at $x = 0$** is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

This series is also called the **Maclaurin series generated by f**.
The partial sum

$$P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

is the **Taylor polynomial or order n for f at $x = 0$**.

Note:

1. $f^{(0)} = f$.

2. Every power series constructed like this converges to f at $x = 0$.

3. The convergence may extend to an interval containing 0. The Taylor polynomials are good approximations near 0.

Example: Find the fourth order Taylor polynomial centered at $x = 0$ that approximates each function .

(a) $f(x) = \sin(2x)$

(b) $f(x) = \sin x^2$

Note: We are not restricted to centered at $x = 0$.

Definition

Let f be a function with derivatives of all orders throughout some open interval containing a. The **Taylor series generated by f at $x = a$** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The partial sum

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the **Taylor polynomial of order n for f at $x = a$**.

Example: Find the Taylor series for $f(x)$ centered at the given value.

(a) $f(x) = 1 + x - x^2$ $\quad x = 2$.

(b) $f(x) = e^x$ $\quad x = 3$.

(c) $f(x) = \sqrt{x}$ $\quad x = 4$.

Note: Taylor series can be added, subtracted, multiplied by constants and powers of x (on the interval of convergence), and the results are Taylor series.

Some common Maclaurin series.

$$\begin{aligned}
\frac{1}{1-x} &= 1+x+x^2+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n \quad (|x|<1)\\
\frac{1}{1+x} &= 1-x+x^2-\cdots+(-x)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n \quad (|x|<1)\\
e^x &= 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{(all real } x\text{)}\\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \quad \text{(all real } x\text{)}\\
\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{(all real } x\text{)}\\
\ln(1+x) &= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad (-1<x\le 1)\\
\tan^{-1}x &= x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}
\end{aligned}$$

Example: Find the Maclaurin series for each function.

(a) $f(x) = x\tan^{-1}x$.

(b) $f(x) = x^2e^{-x}$.

(c) $f(x) = e^{-x^2}\cos x$.

Example:

(a) Find a power series to represent $f(x) = \dfrac{\sin x}{x}$.

(b) The power series in (a) is not really a Maclaurin series for f, because f is not eligible to have a Maclaurin series. Why not?

(c) If we redefine f as follows, then the power series in (a) will be a Maclaurin series for f. What is the value of k?

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & x\neq 0\\ k & x=0\end{cases}$$

Taylor's Theorem

Remarks:

1. We use nth order Taylor polynomials to approximate functions, since we have to deal with *finite* sums.

2. We need to know how good an approximation we have.

Some problems to think about.

1. Find a Taylor polynomial that will serve as a reasonable approximation for $\sin x$ on the interval $[-\pi, \pi]$.

2. Find a formula for the error if we use $1 + x^2 + x^4 + x^6$ to approximate $1/(1 - x^2)$ over the interval $(-1, 1)$.

Theorem

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Note:

1. This theorem provides a formula for the polynomial approximation and for the error involved in using this approximation (over the interval I).

2. $R_n(x)$ is the **remainder of order n**, or just the **error term**.

   It is also called the **Lagrange form** of the remainder. The bounds on the error are therefore called **Lagrange error bounds**.

3. If $R_n(x) \to 0$ as $n \to \infty$ for all x in I, then the Taylor series generated by f at $x = a$ **converges to f** on I.

Example: Prove that the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

converges to $\cos x$ for all real x.

Theorem

If there are positive constants M and r such that $|f^{(n+1)}(t)| \leq M r^{n+1}$ for all t between a and x, then the remainder $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}.$$

If these conditions hold for every n and all the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Example: Use the previous theorem to show that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges to e^x for all real x.

Example: The approximation $\sqrt{1+x} \approx 1 + (x/2)$ is used when x is small. Estimate the maximum error when $|x| < 0.01$.

Radius of Convergence

Background:

1. We need a strategy for finding the interval of convergence of an arbitrary power series.

2. Any power series (centered at $x = a$) always converges at $x = a$.

3. Some power series converge for all real numbers, and some converge on a finite interval centered at a.

Theorem

There are three possibilities for $\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence:

1. There is a positive number R such that the series diverges for $|x-a| > R$ but converges for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

Remarks:

1. The number R is the **radius of convergence**.

   The set of all values of x for which the series converges is the **interval of convergence**.

2. If $0 < R < \infty$ there is still a convergence question at the endpoints of the interval.

3. We need to learn how to find the radius of convergence, and then worry about the endpoints.

Here are some important definitions and theorems.

Theorem

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem

The Direct Comparison Test

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series $\sum d_n$ of nonnegative terms with $a_n \geq d_n$ for all $n > N$, for some integer N.

Definition

Absolute Convergence

If the series $\sum |a_n|$ of absolute values converges, then $\sum a_n$ **converges absolutely**.

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Theorem

The Ratio Test

Let $\sum a_n$ be a series with positive terms, and with $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$. Then,

(a) The series converges if $L < 1$.

(b) The series diverges if $L > 1$.

(c) The test is inconclusive if $L = 1$.

Example: Determine whether each series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} n^2 e^{-n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$

Example: Find the radius of convergence for each power series.

(a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

Example: Find the interval of convergence of the series and within this interval find the sum of the series as a function of x.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2}-1\right)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x^2-1}{3}\right)^n$

Example: Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges and find the sum.

Convergence at the Endpoints

Background:

1. In general, it's hard to find the exact sum of a series.

   Geometric series are pretty easy, and there are standard techniques for telescoping sums.

   But usually, $\lim_{n\to\infty} s_n = ?$

2. We need some tests to determine convergence or divergence (without actually finding the sum).

> **Theorem**
>
> The Integral Test
>
> Let $\{a_n\}$ be a sequence of positive terms. Suppose $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

Example: Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

> ***p*-Series**
>
> The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Note:

1. If $p = 1$, this is the harmonic series.

2. We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{but} \qquad \int_1^{\infty} \frac{1}{x^2}\, dx = 1$$

Example: Determine whether the series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Theorem

The Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N a positive integer).

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example: Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$

A series in which the terms are alternately positive and negative is an **alternating series**.

Some examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n 4}{2^n}$$

Theorem

The Alternating Series Test

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. Each u_n is positive;
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N;
3. $\lim_{n\to\infty} u_n = 0$.

Example: Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$

Note: If an alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions in the previous theorem, then the truncation error for the nth partial sum is less than u_{n+1} and has the same sign as the first unused term.

Example: Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ correct to three decimal places.

Definition

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Definition

A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

Rearrangements of Absolutely Convergent Series

If $\sum a_n$ converges absolutely, and if $b_1, b_2, b_3, \ldots, b_n, \ldots$ is any rearrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Rearrangements of Conditionally Convergent Series

If $\sum a_n$ converges conditionally, then the terms can be arranged to form a divergent series. The terms can also be rearranged to form a series that converges to *any* preassigned sum.

Testing a Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$ for Convergence

1. Use the Ratio Test to find the values of x for which the series converges absolutely. Ordinarily, this is an open interval
$$a - R < x < a + R.$$
In some cases, the series converges for all values of x. The series may converge only at $x = a$.

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. The Ratio Test fails at these points. Use a comparison test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a - R < x < a + R$, conclude that the series diverges (it does not even converge conditionally) for $|x - a| > R$, because for those values of x, the nth term does not approach zero.

Example: For what values of x do the following series converge?

(a) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

(b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

(c) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

(d) $\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$