Department of Mathematics, Computer Science, and Statistics
Bloomsburg University
Bloomsburg, Pennsylvania 17815

L'Hôpital's Rule

Summary

Many limits may be determined by direct substitution, using a geometric argument, or some algebraic technique. However, various indeterminate forms, particularly those involving both algebraic and transcendental functions, require L'Hôpital's Rule. This presentation will focus on motivating and teaching L'Hôpital's Rule, AP type problems involving several indeterminate forms, numerical and graphical confirmation, and some interesting applications.

Address:

Stephen Kokoska
Department of Mathematics,
Computer Science, and Statistics
Bloomsburg University
400 East Second Street
Bloomsburg, PA 17815

Phone: (570) 389-4629
Email: skokoska@planetx.bloomu.edu

Background:

1. When solving previous limit problems we saw the indeterminate forms $0/0$ and ∞/∞. (There are other indeterminate forms.)

2. Various algebraic techniques were used.

   Example: $\displaystyle\lim_{x\to 1} \frac{x^2 - x}{x^2 - 1} =$

   Example: $\displaystyle\lim_{x\to\infty} \frac{x^3 - 1}{5x^3 - 2x + 7} =$

3. Consider a procedure to solve problems like:

   (a) $\displaystyle\lim_{x\to 0} \frac{2^x - 1}{x}$

   (b) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x - 1}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

4. To solve these problems we use L'Hôpital's Rule.

L'Hôpital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $0/0$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Note:

1. In words: The limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided the conditions are satisfied.

   It is important to verify the conditions regarding the limits of f and g before using L'Hôpital's Rule.

2. L'Hôpital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity.

   That is, $x \to a$ can be replaced by any of the following symbols.
   $x \to a^+, \quad x \to a^-, \quad x \to \infty, \quad x \to -\infty.$

3. For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, consider the following proof.

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x \to a} \frac{f(x)}{g(x)}$$

4. Some graphical evidence:

   (a) The first graph shows two differentiable functions f and g. Each approaches 0 as $x \to a$.

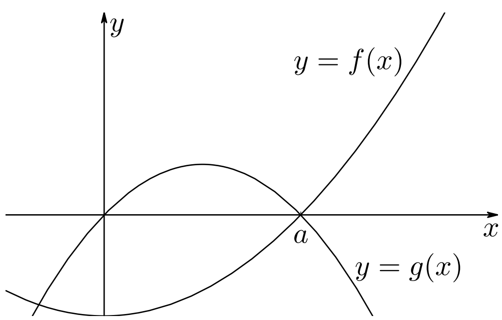
   (b) If we zoom in around $(a, 0)$, the graphs become locally linear.

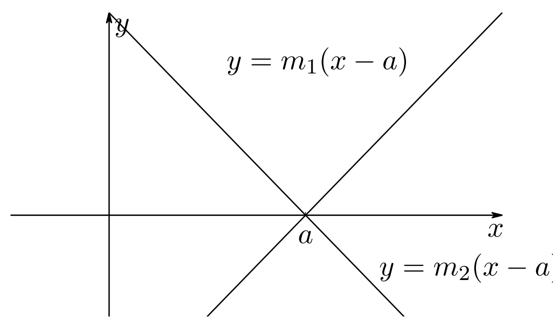
   If the functions were actually linear, then their ratio would be

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

   which is the ratio of their derivatives.

   This suggests that $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$





5. The first appearance in print of l'Hôpital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hôpital in 1696. The example used to illustrate the rule was

$$\lim_{x \to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} \quad \text{where } a > 0.$$

Example: $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

Example: $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x^2}$

Example: $\lim\limits_{x\to \pi^-} \dfrac{\sin x}{1-\cos x}$

Example: $\lim_{x \to 0} \frac{2^x - 1}{x}$

Example: $\lim_{x \to \infty} \frac{x}{e^x}$

Example: $\lim_{x \to \infty} \frac{e^x}{x}$

Example: $\lim_{x \to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}$

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then the limit

$$\lim_{x\to a}[f(x)g(x)]$$

is an indeterminate form of type $0 \cdot \infty$.

This limit may be 0, ∞, or some compromise, a finite nonzero number.

To solve this problem, write the product as a quotient;

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)}$$

The quotient will then have indeterminate form $0/0$ or ∞/∞.

Example: $\lim_{x\to 0^+} x^2 \ln x$

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}[f(x) - g(x)]$$

is an indeterminate form of type $\infty - \infty$.

This limit is a contest between f and g.
If f wins, the limit is ∞. if g wins, the limit is $-\infty$.
Or, there may be some finite number compromise.

To determine this limit, convert the difference into a quotient: common denominator, rationalization, factoring.

The quotient will then have indeterminate form $0/0$ or ∞/∞.

Indeterminate Powers

There are several indeterminate forms that arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be solved by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x)\ln f(x).$$

This will lead to an indeterminate product.

Find the limit of $\ln y = L$. Then $\lim_{x\to a} y = e^L$.

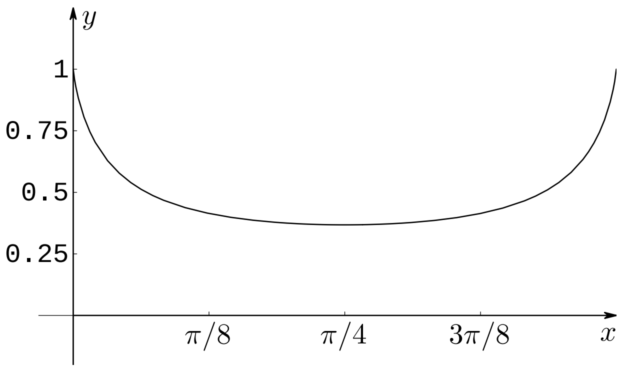
Example: $\lim_{x \to 0^+} (1 - 2x)^{1/x}$

Example: $\lim_{x \to \infty} x^{1/x}$

Example: $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

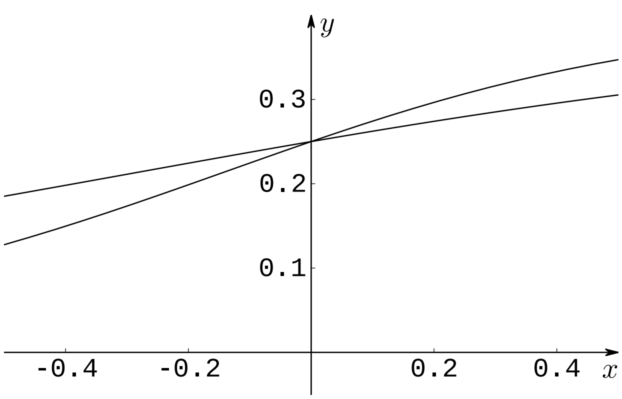
Example: Consider $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

Use a graph to estimate the value of the limit. Use L'Hôpital's Rule to find the exact value.



Example: Let $f(x) = e^x - 1$ and $g(x) = x^3 + 4x$.

Graph both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$. Find $\lim_{x \to 0} \dfrac{f(x)}{g(x)}$.



Example: If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

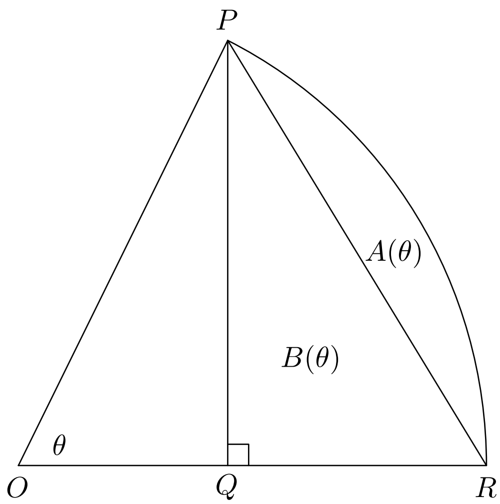
where g is the acceleration due to gravity and c is a positive constant.

(a) Find $\lim_{t \to \infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hôpital's Rule to calculate $\lim_{m \to \infty} v$. What can you conclude about the speed of a very heavy falling object?

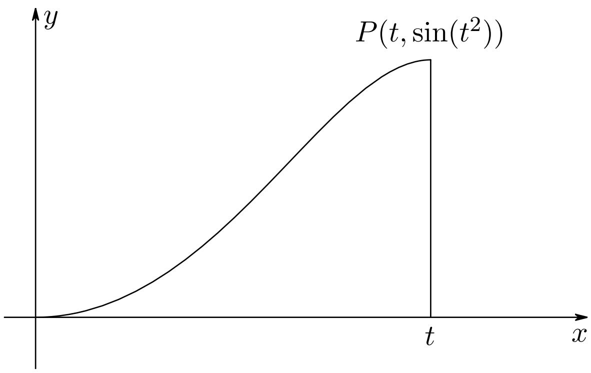
Example: Let $S(x) = \int_0^x \sin(\pi t^2/2)\, dt$ Evaluate $\lim_{x \to 0} \frac{S(x)}{x^3}$.

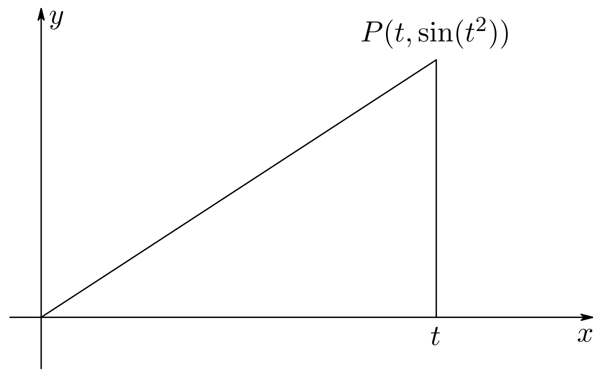
Example: The figure below shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim\limits_{\theta \to 0^+} \dfrac{A(\theta)}{B(\theta)}$.



Example: Let $A(t)$ be the area under the curve $y = \sin(x^2)$ from 0 to t.
Let $B(t)$ be the area of the triangle with vertices O, P, and $(t, 0)$.

Find $\lim_{t \to 0^+} \dfrac{A(t)}{B(t)}$.





Example: Suppose a and b are positive real numbers. Find $\lim\limits_{x\to\infty}\left(\dfrac{a^{1/x}+b^{1/x}}{2}\right)^{x}$.

Example: A weight of mass m is attached to a spring suspended from a support. The weight is set in motion by moving the support up and down according to the formula $h = A\cos\omega t$, where A and ω are positive constants and t is time. If frictional forces are negligible, then the displacement s of the weight from its initial position at time t is given be

$$s = \frac{A\omega^2}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$ for some constant k and with $\omega \neq \omega_0$. Find $\lim\limits_{\omega \to \omega_0} s$.

This shows the resulting oscillations increase in magnitude and is an example of the phenomenon of resonance.

Example: Suppose that n is a fixed positive integer. Find

$$\lim_{x \to \infty} \left(\sqrt[n]{x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0} - x \right)$$

Example: The surface area of an ellipsoid obtained by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a > b)$ around the x-axis is

$$A = 2\pi ab \left[\frac{b}{a} + \frac{a}{c} \sin^{-1}\left(\frac{c}{a}\right) \right]$$

where $c^2 = a^2 - b^2$. Find $\lim_{b \to a} A$.

Example: Find all positive integers n for which the given limit exists.

$$\lim_{x \to 1^+} \frac{e^{x-1} - x}{(x-1)^n}$$