Department of Mathematics, Computer Science, and Statistics
Bloomsburg University
Bloomsburg, Pennsylvania 17815

Pre-AP Strategies in Mathematics: Functions

Summary

This workshop is designed to address content, pedagogy, and assessment. Emphasis is placed on the concept of a function as it is developed in the mathematics curriculum in grades 6-12. In pedagogy, the workshop illustrates a guided exploration approach that emphasizes student thinking as the key to learning and student communication as the key to assessing student learning. In assessment, the workshop demonstrates the importance of embedding assessment within instruction to monitor the mathematical growth and development of the student. This workshop is for teachers, but the focus is on students, discussion of content as well as pedagogical and assessment practices directed to improving student success in learning mathematics.

Address:

Stephen Kokoska
Department of Mathematics,
Computer Science, and Statistics
Bloomsburg University
400 East Second Street
Bloomsburg, PA 17815

Phone: (570) 389-4629
Email: skokoska@bloomu.edu

Pre-AP Strategies in Mathematics - Function

Workshop Goals

1. To understand the concept of function as well as its place and its development from Middle School Mathematics up to and including Calculus.

2. To see the vertical curriculum development of basic middle school mathematics, such as patterns and pattern recognition, and its relationship to the Calculus concept of accumulation.

3. To demonstrate a Guided-Exploration pedagogy that fosters thinking in mathematics and provides all students with a learning-by-doing classroom environment.

4. To emphasize the value of Multiple Representations as a pedagogical process for helping students learn mathematics.

5. To gain a deeper understanding of:

   (a) the concept of function;

   (b) how the concept of function differs from relation;

   (c) function as a mathematical model of real-world phenomena;

   (d) the critical place of linear function in the mathematics curriculum;

   (e) the role of multiple representations as a means of understanding, analyzing, interpreting, and constructing functions and function models of real-world phenomena.

Make an Input-Output Rule

Input, x	**Output**, y
0	
1	
2	
3	
4	
5	

Grade	Concept of Function: Formal, Informal, or Not at all?	Describe or define the function concept	Prerequisite Knowledge
6			
7			
8			
9			
10			
11			
12			

Some function definitions

1. A function is a curve or a graph.

2. A function is something that is of the form $y =$ an algebraic expression in x.

3. A function is an equation in variables x and y.

4. A function is an algebraic rule in x that assigns a value, called y, to a value x.

5. A function is something of the form $y = f(x)$.

6. A function is a rule that assigns a value, called y, to a value, called x.

7. The word *function* indicates that the value of one variable depends on the value of another variable.

8. A function is a relationship between two quantities in which the value of one quantity, the dependent variable, is uniquely determined by the value of the other quantity, the independent variable.

In *Calculus* by Howard Anton, Irl Bivens, Stephen Davis, 7th Edition Update, ISBN: 0-471-43312-8, John Wiley & Sons, Inc., 2003, there are two definitions given:

(a) If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that y is a function of x.

(b) A function f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$.

In *Calculus* by James Stewart, 5th Edition, Brooks/Cole, 2003. we have the following definition:

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Grade	Function Type	Function Features, Properties, Characteristics
6		
7		
8		
9		
10		
11		
12		

Find the value of $|2 - \sqrt{5}|$.

What is the value of $| - a|$?

Pre-Algebra

Ratio, Direct Variation:

$y = kx$

$d = rt$

In Middle School Mathematics, Students are introduced to:

Square Root: $\sqrt{x}$

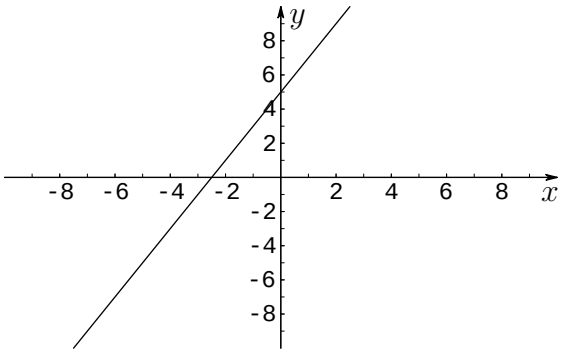
Absolute Value: $|x|$

These are understood only as numerical operations rather than a relationship of input and output.

Algebra I

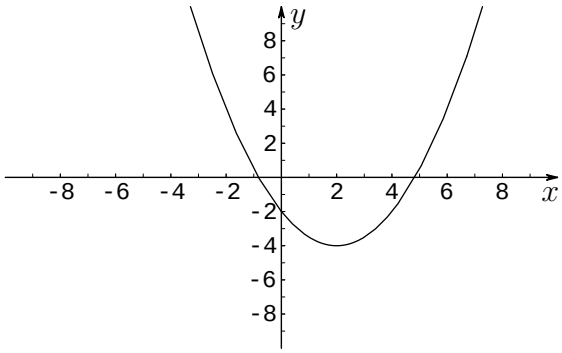
Linear Function:

$y = mx + b$



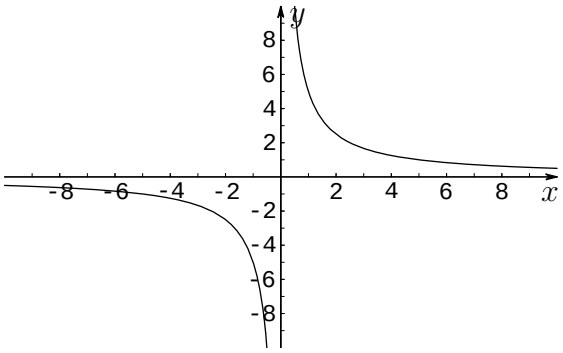
Quadratic Function:

$y = ax^2 + bx + c$



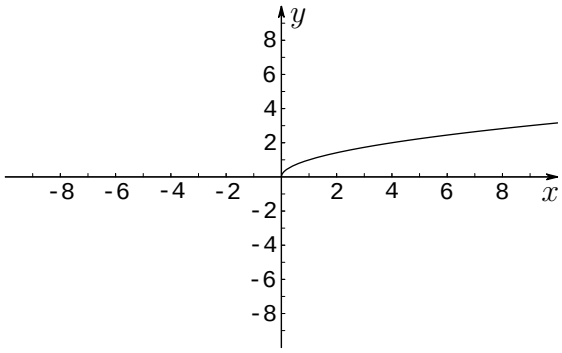
Inverse Variation:

$y = k/x$



Square Root:

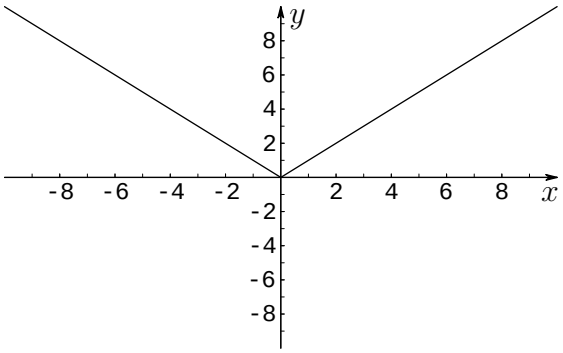
$y = \sqrt{x}$



Algebra II

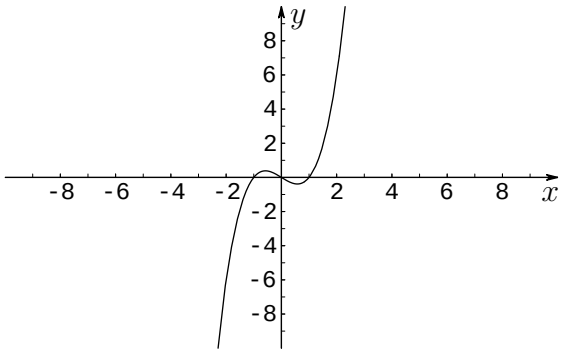
Absolute Value Function

$y = |x|$



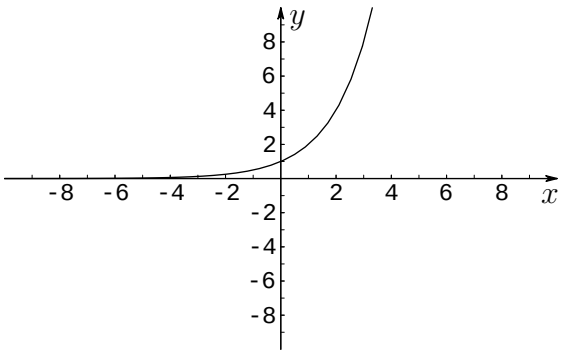
Polynomial

$y = x^3 - x$



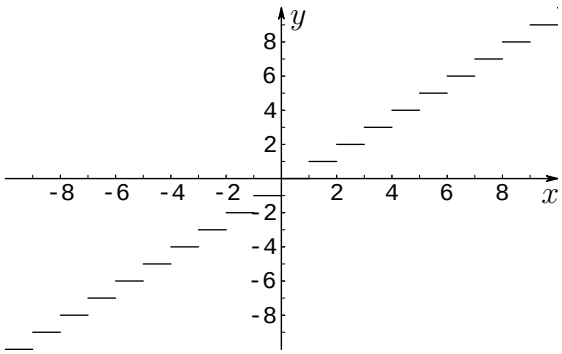
Exponential Function

$y = 2^x$



Step Function

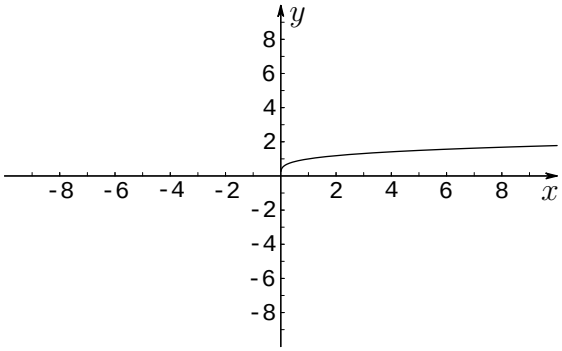
$y = [\![x]\!]$



Algebra II

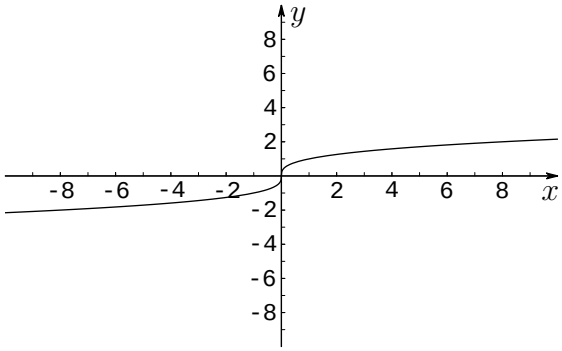
Even Root Function

$y = \sqrt[4]{x}$



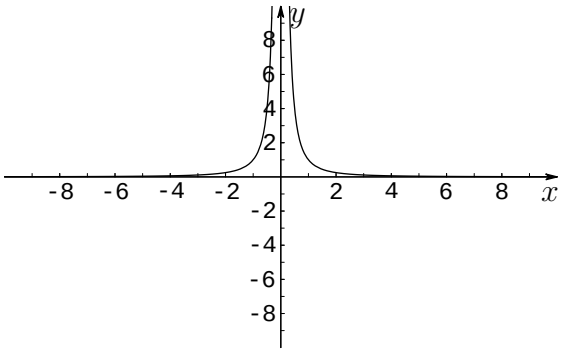
Inverse Function

$y = \sqrt[3]{x}$

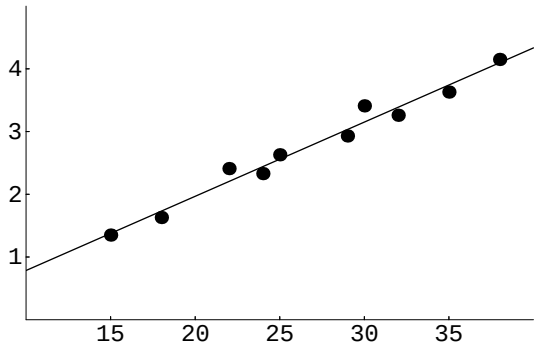


Special Functions

$y = 1/x^2$



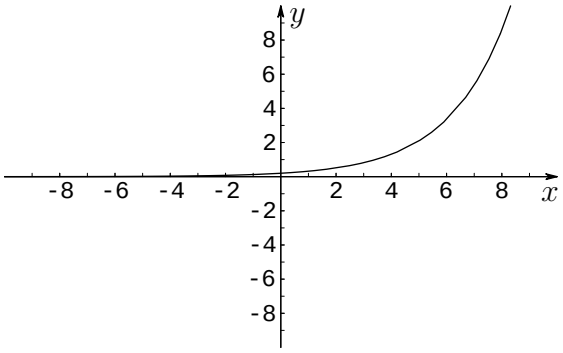
Trend Lines



Pre-Calculus

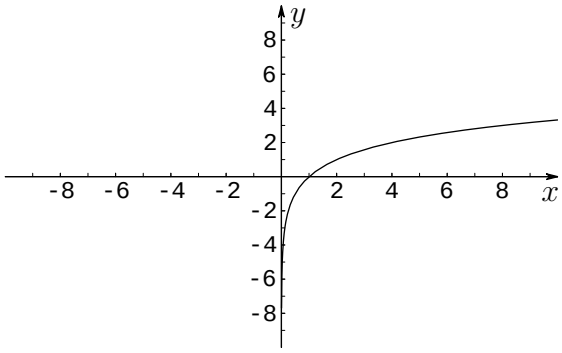
Exponential Function

$y = a(b^x)$



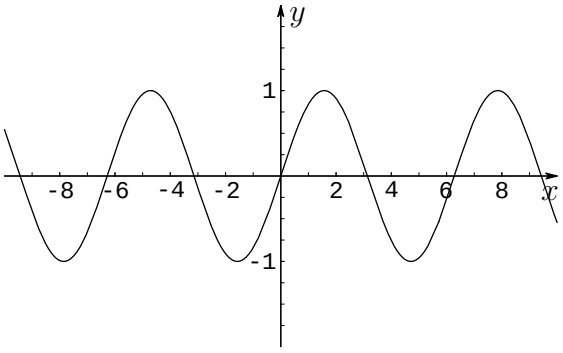
Logarithmic Function

$y = \log_b x$



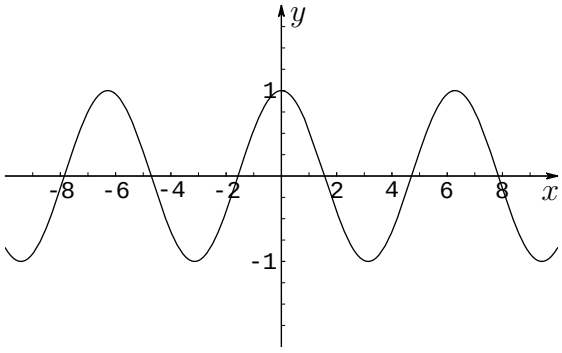
Trig Function

$y = \sin x$



Trig Function

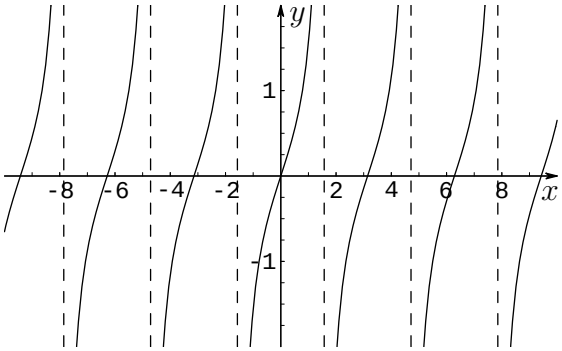
$y = \cos x$



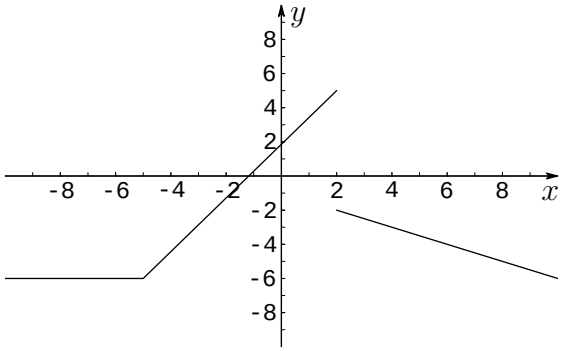
Pre-Calculus

Trig Function

$y = \tan x$

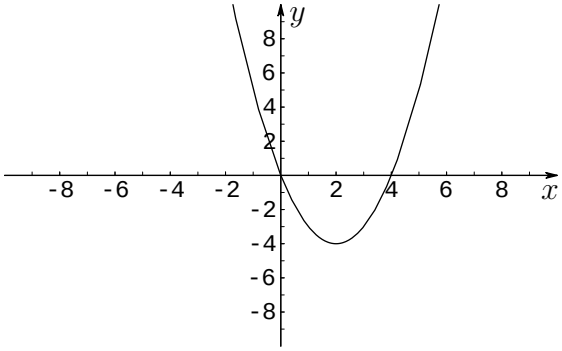


Piecewise Linear Function



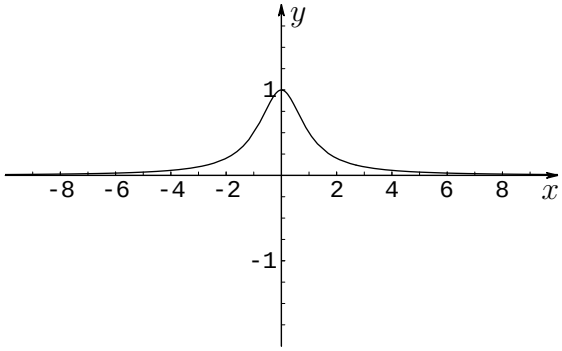
Function Transformations

$y = (x-2)^2 - 4$



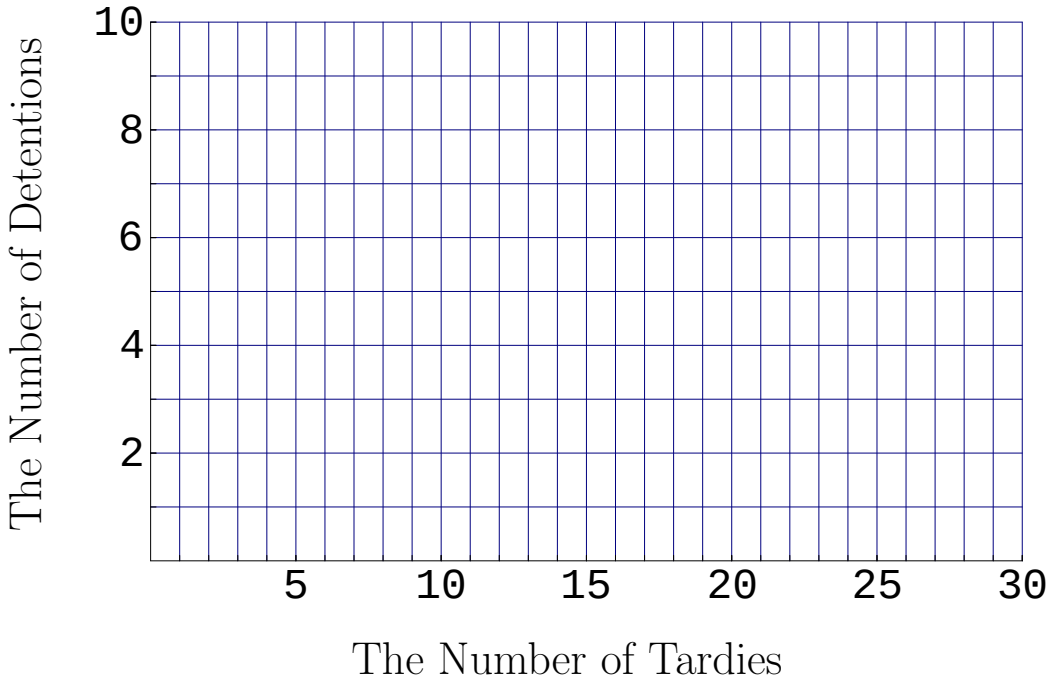
Function Composition

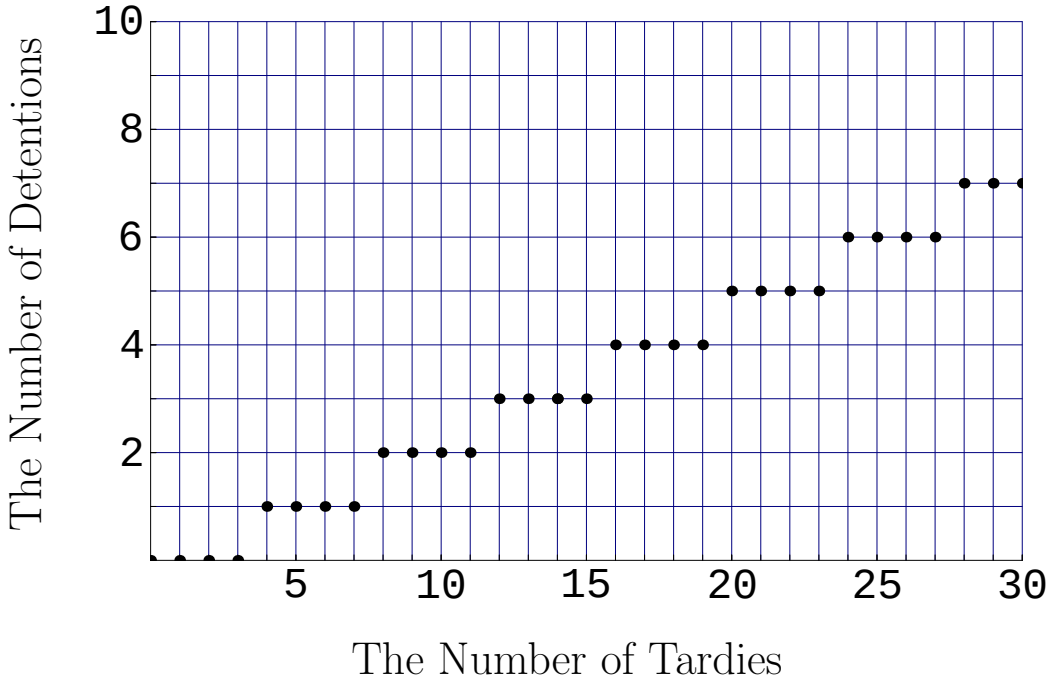
$y = \dfrac{1}{1+x^2}$



Tardies and Detentions

Number of Tardies	Total Number of Detentions
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	


The Number of Detentions
10
8
6
4
2
5
10
15
20
25
30
The Number of Tardies


The Number of Detentions
10
8
6
4
2
5
10
15
20
25
30
The Number of Tardies

Video / DVD Rentals

Regular Plan	
Videos rented	Cost (in dollars)
0	
1	
2	
3	
4	
5	

Membership Plan	
Videos rented	Cost (in dollars)
0	
1	
2	
3	
4	
5	

Typical Student Responses: Regular Plan

1. The cost is equal to \$2 multiplied by the number of videos.
2. Every video/DVD rented costs 2 more dollars.
3. $2x$ or $x \cdot 2$.
4. Videos rented times 2.
5. Add by 2; Counting by 2.
6. $3 \cdot 2 = 6$
7. $n \cdot 2$, or some other letter for the number of videos rented.
8. The cost values are multiples of 2.
9. $2 + 2 = 4$, $4 + 2 = 6$, $6 + 2 = 8$, ...
10. Pattern using and going by 2 using multiplication.
11. If you divide number of videos into cost you'll always get the video rental charge.

Typical Student Responses: Membership Plan

1. Start with 40, then add 0.75 to the previous cost in the table to get the next cost in the table.

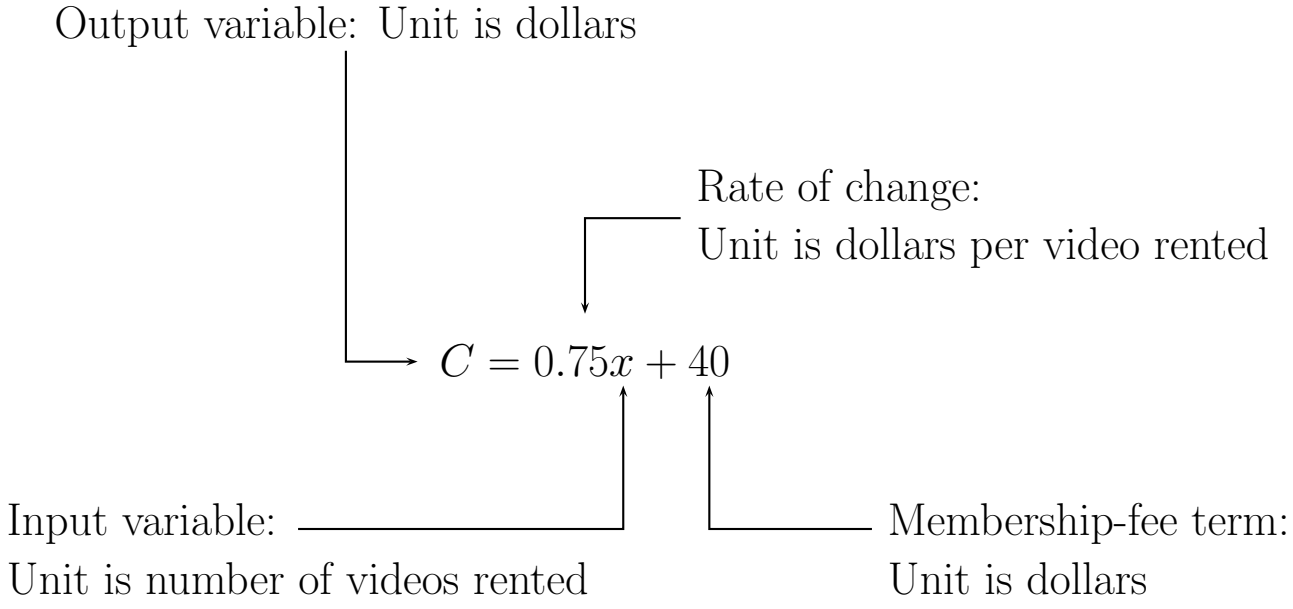
2. Start with \$40 then add \$0.75.

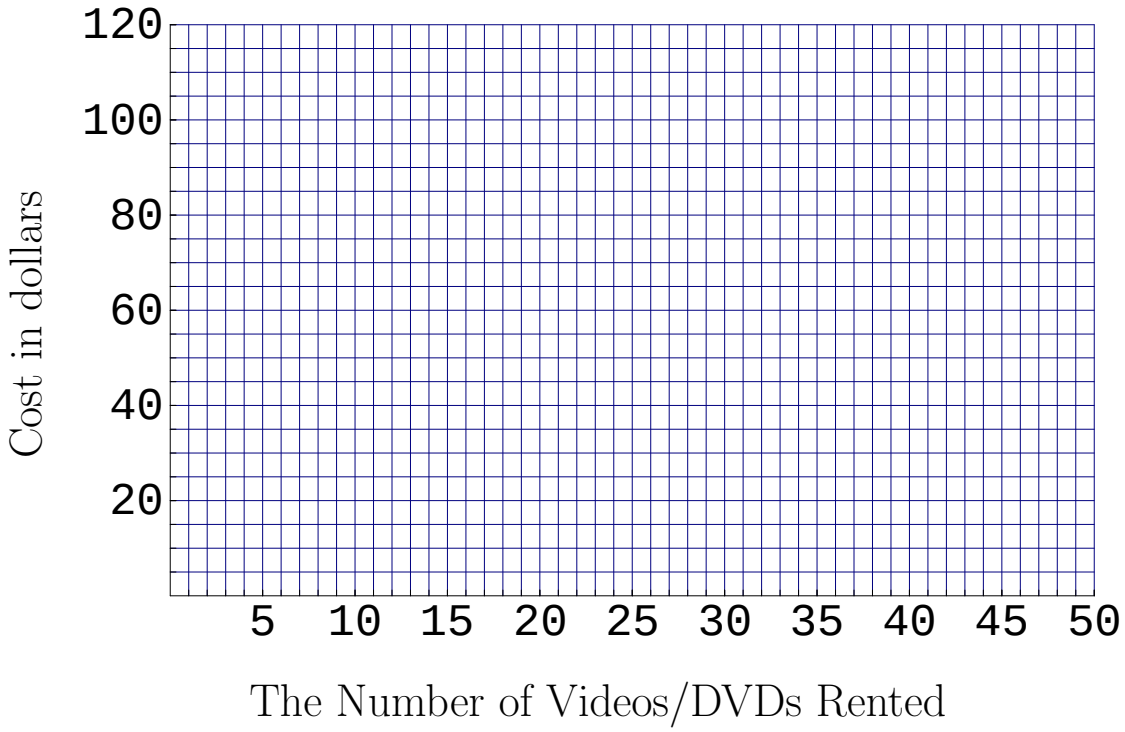
3. $40 + x \times 0.75$.

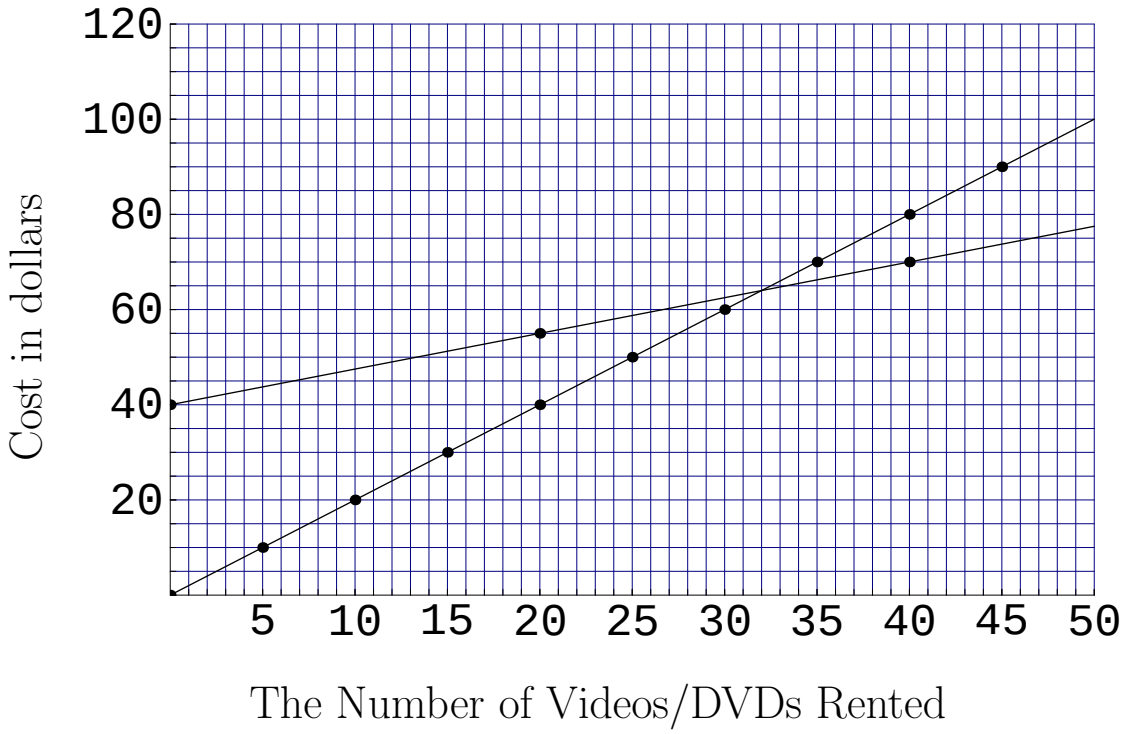
4. $x \times 0.75$ (of some other letter for the number of videos rented).

5. $3 \cdot 0.75 + 40 = 42.25$.

6. Add 0.75 to the previous cost in the table to get the next cost in the table.


Output variable: Unit is dollars
Rate of change:
Unit is dollars per video rented
$C = 0.75x + 40$
Input variable:
Unit is number of videos rented
Membership-fee term:
Unit is dollars


120
100
80
60
40
20
Cost in dollars
5 10 15 20 25 30 35 40 45 50
The Number of Videos/DVDs Rented


120
100
80
60
40
20
Cost in dollars
5 10 15 20 25 30 35 40 45 50
The Number of Videos/DVDs Rented

_____ *The Oscar* Regular Plan: $5 per video DVD rental

_____ *Reruns* Regular Plan: $6 per video/DVD rental

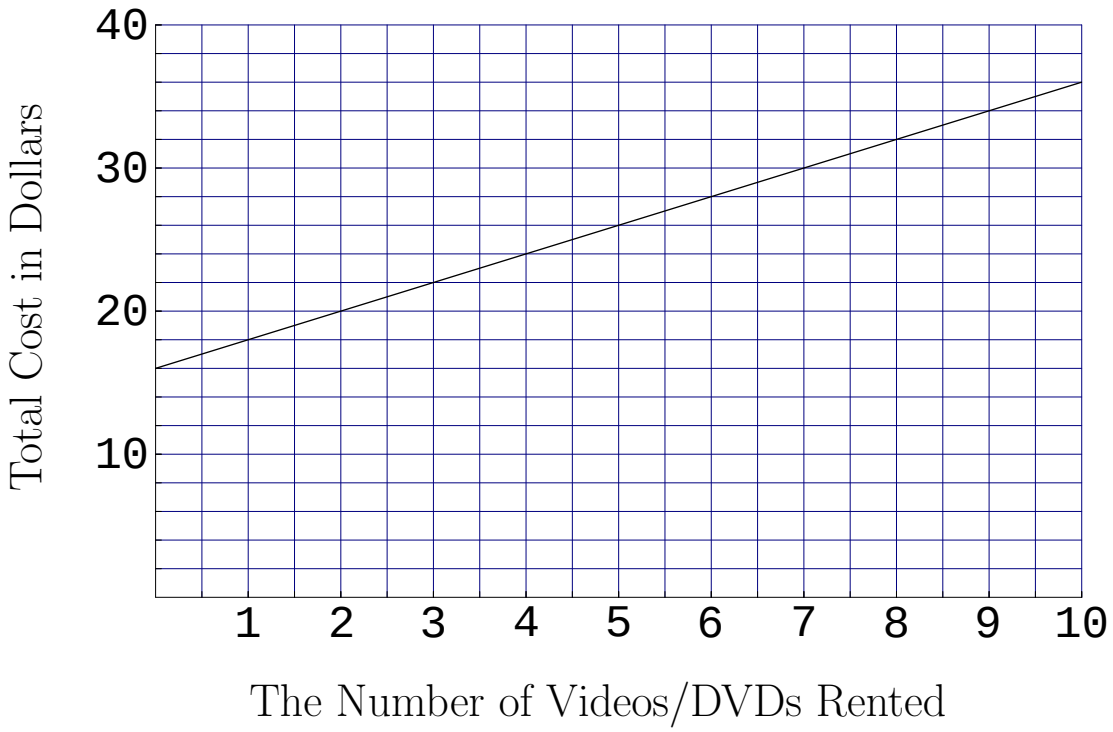
_____ *Reruns Video Club Plan*: $25 annual membership plus $3 per video rental

_____ *The Oscar* Video Club Plan: $50 annual membership plus $2 per video rental

Two video/DVD rental clubs offer two different rental fee plans:

1. Club A has a \$8 membership fee and charges \$4 for each rented video.

2. The Club B plan is given by the graph below which represents the total cost of renting videos from Club B.



Cost of Pizza

The following table is taken from the *To Go* menu at Hideaway's Pizzeria.

Items	Cost (in dollars)
0	3.20
1	3.50
2	3.80
3	4.10
4	4.40

Recognizing Linear Data in a Table

Examine the following tables of data. Is this data linear data? If it is, then find a rule for it in the form $y = mx + b$.

1. Table 1

x	-5	-2	1	4	7	10
y	10	19	28	38	47	56

Linear: Yes or No Linear rule: _____

2. Table 2

x	-15	-9	5	6.5	11.6	16
y	101	81.8	37	32.2	15.88	1.8

Linear: Yes or No Linear rule: _____

The Mackinac Island Bicycle Rental Shop

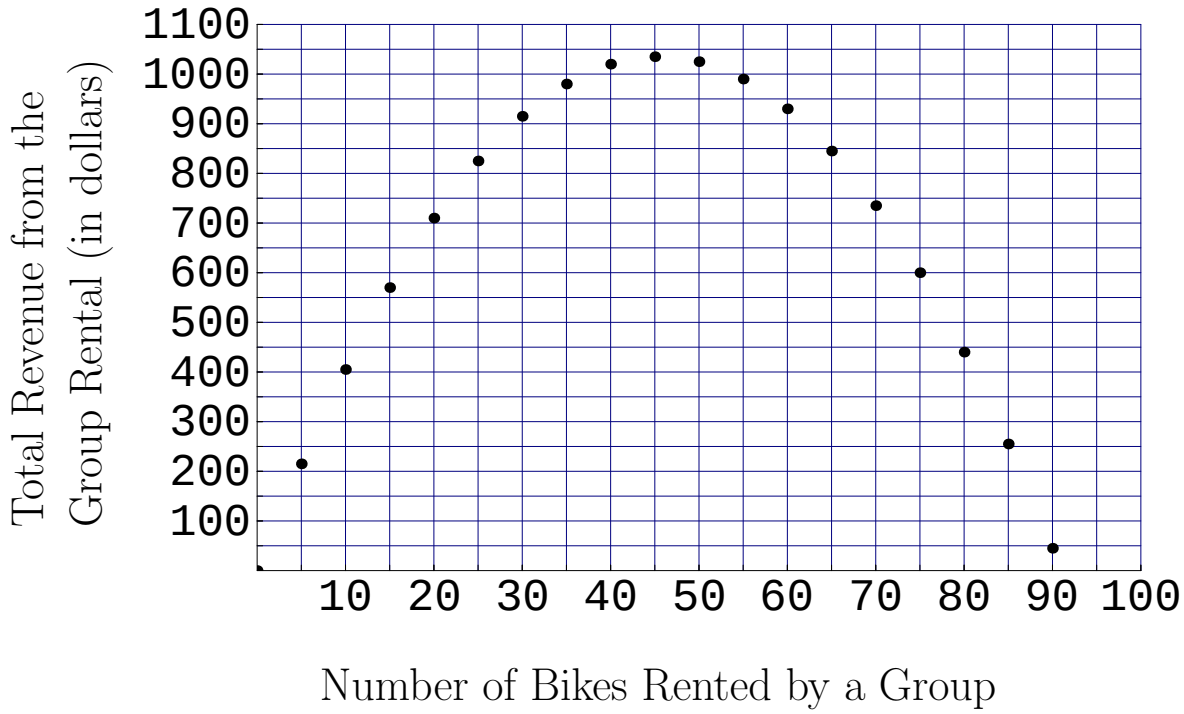
Number of Bikes Rented by a Group	Daily Bike Rental Charge (in dollars per day)
1	45.00
2	44.50
3	44.00
4	43.50
5	43.00

$B(n) =$ the daily rental charge per bike when a group rents n bikes.

Total Revenue

Total Revenue for Group Bike Rentals		
Number of Bikes Rented by a Group n	Daily Bike Rental Charge $B(n)$	Total Revenue $R(n)$
1		
2		
3		
4		
5		

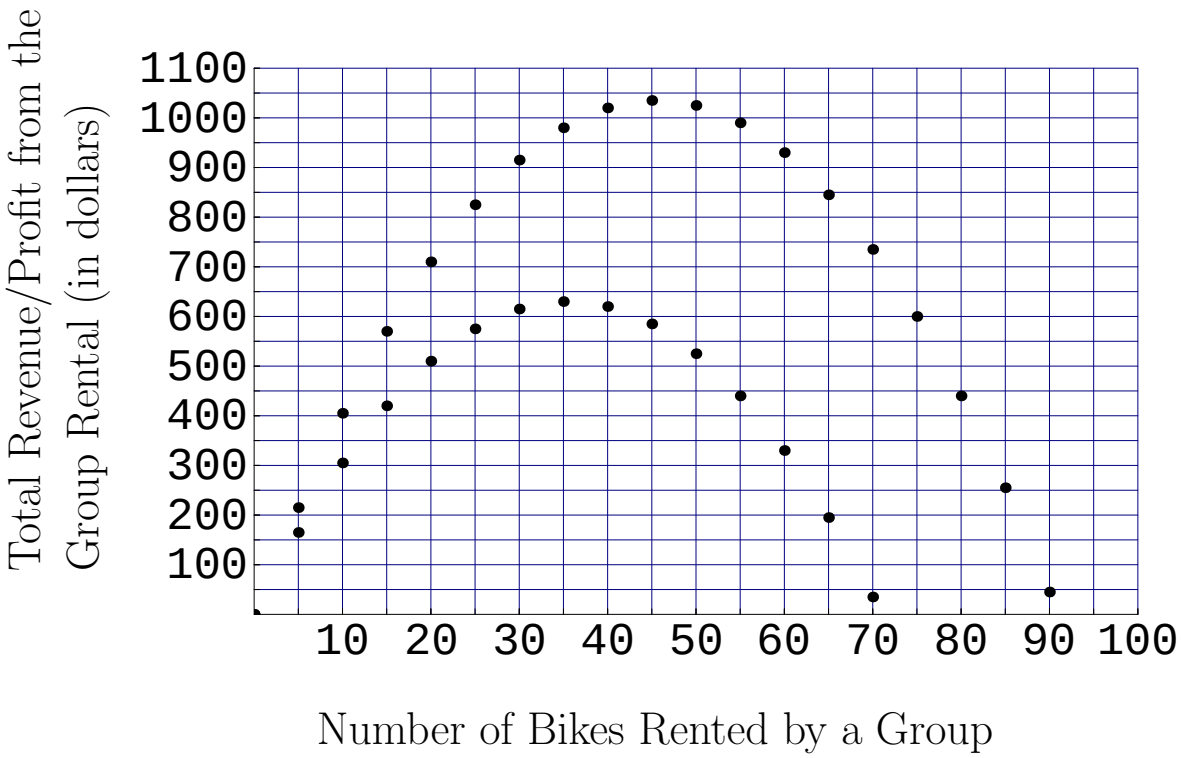
$R(n)$ = the total revenue, in dollars, when a group of size n, rents bikes from the Bicycle Shop.


Total Revenue from the
Group Rental (in dollars)
1100
1000
900
800
700
600
500
400
300
200
100
10 20 30 40 50 60 70 80 90 100
Number of Bikes Rented by a Group

Total Profit

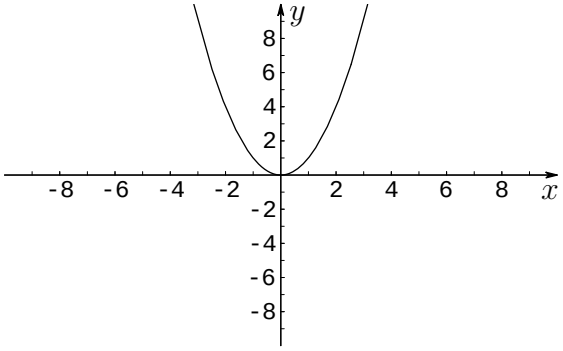
Total Profit for Group Bike Rentals		
Number of Bikes Rented by a Group n	Daily Bike Rental Charge $B(n)$	Total Profit $P(n)$
1		
2		
3		
4		
5		

$P(n) =$ the total profit, in dollars, when a group of size n, rents bikes from the Bicycle Shop.

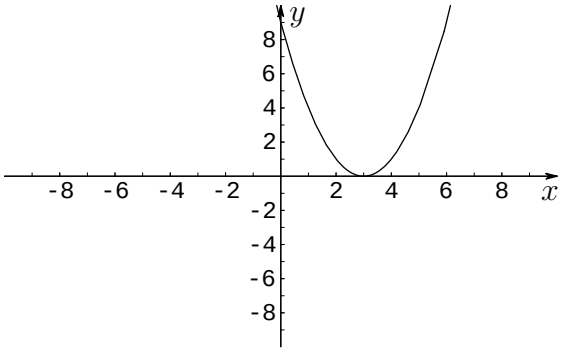

Total Revenue/Profit from the Group Rental (in dollars)
1100
1000
900
800
700
600
500
400
300
200
100
10 20 30 40 50 60 70 80 90 100
Number of Bikes Rented by a Group

Function Notation and Transformations

The Parent Function: $f(x) = x^2$

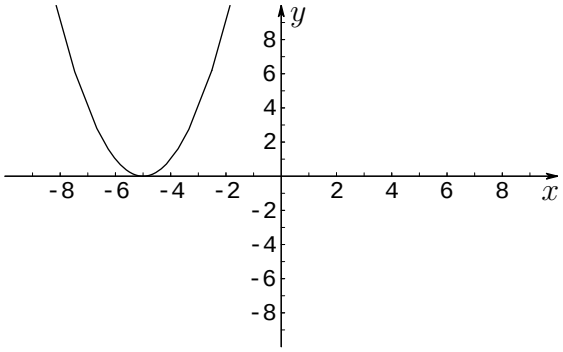


Horizontal Shift to the right



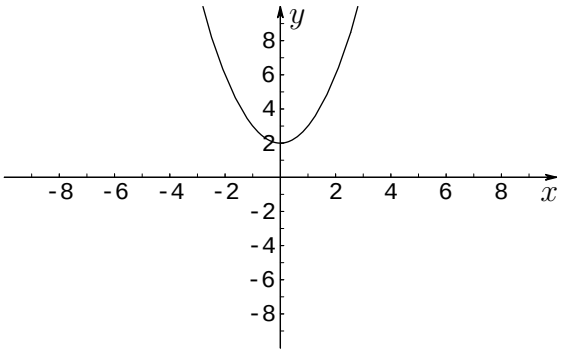
$y = f(x - h),\ h > 0$

Horizontal Shift to the Left



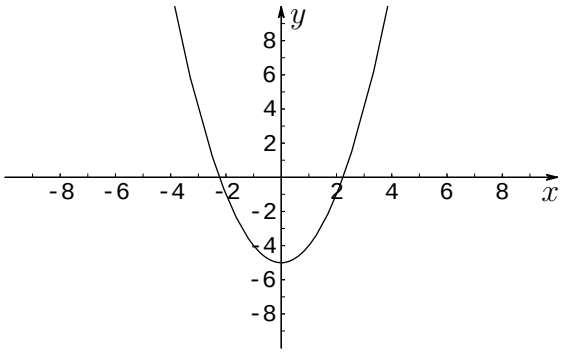
$y = f(x + h),\ h > 0$

Vertical Shift Up



$y = f(x) + k,\ k > 0$

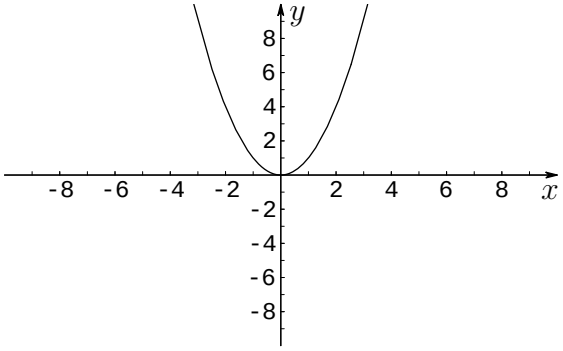
Vertical Shift Down



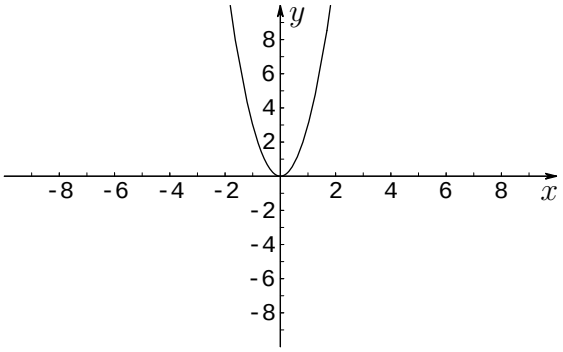
$y = f(x) - k,\ k > 0$

Function Notation and Transformations

The Parent Function: $f(x) = x^2$

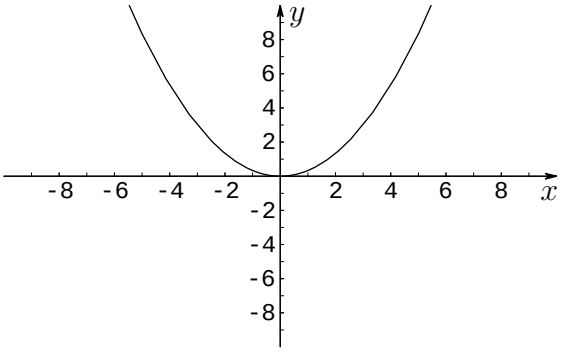


Vertical Stretch



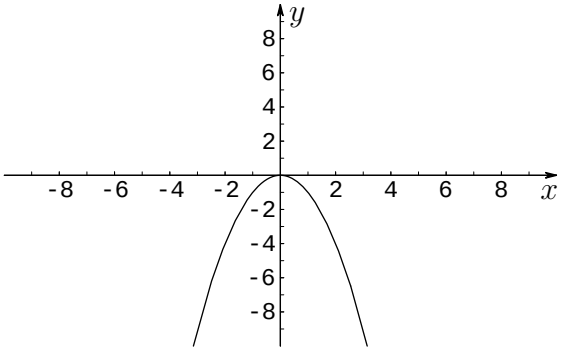
$y = af(x),\ a > 1$

Vertical Shrink



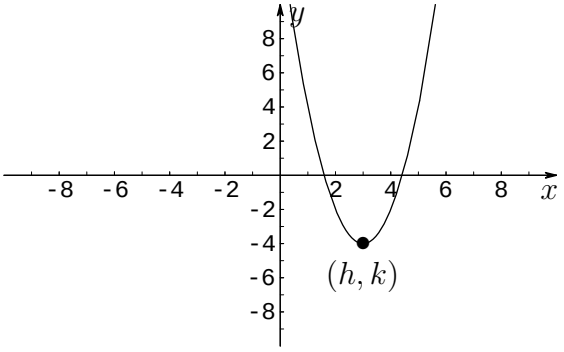
$y = af(x),\ 0 < a < 1$

Reflection Across the x-axis



$y = -f(x)$

Putting It All Together



$y = af(x - h) + k$

Transformations of Functions

Use transformations and the graph of the parent function $f(x) = |x|$ to sketch a graph of the following functions.

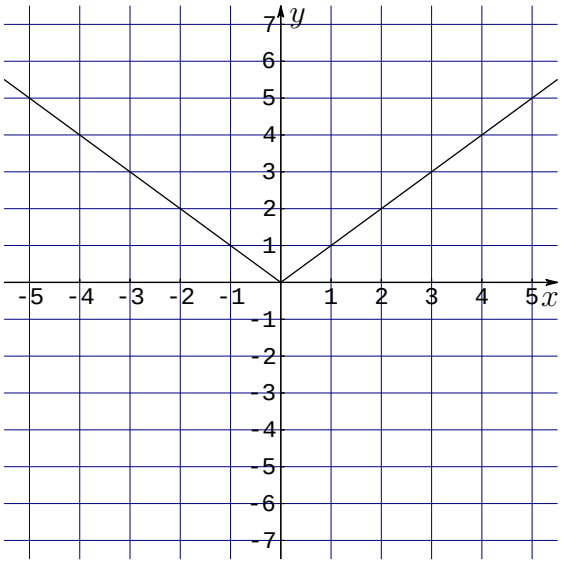
1. $y = -2f(x)$

2. $y = f(x - 3)$

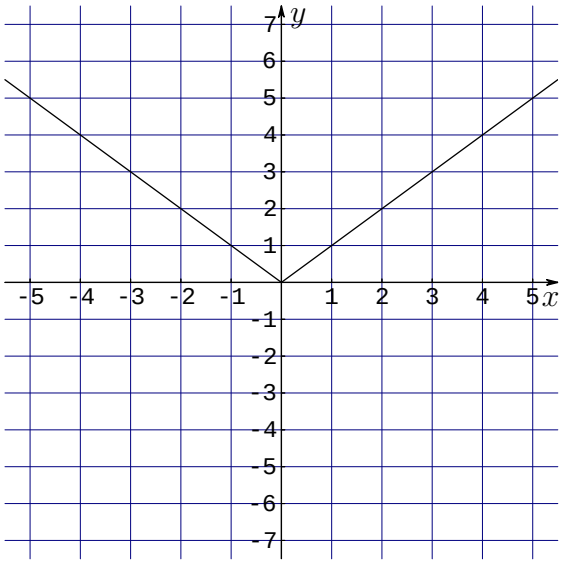
3. $y + 2 = \frac{1}{2}f(x)$

4. $y + 2 = \frac{1}{2}f(x - 1)$

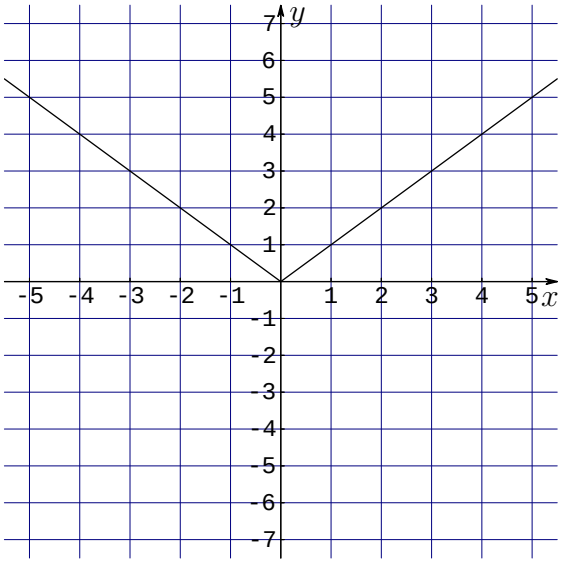
$y = -2f(x)$



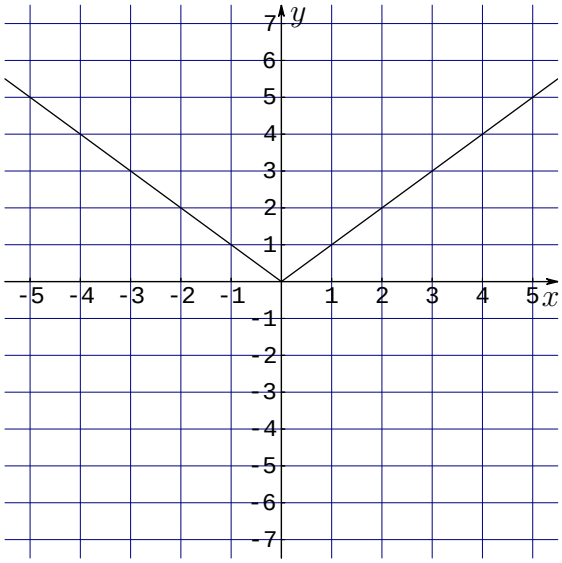
$y = f(x - 3)$



$y = \frac{1}{2}f(x) - 2$



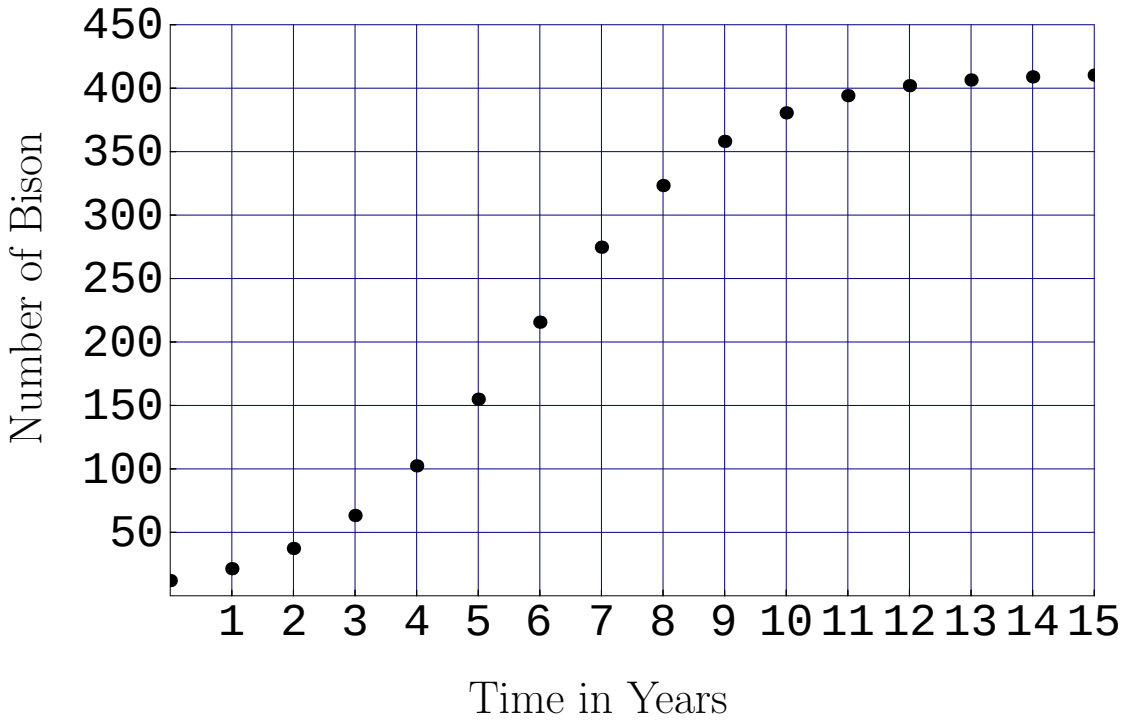
$y = \frac{1}{2}f(x - 1) - 2$



The Bison Population in a Wildlife Reserve

t	0	1	2	3	4	5	6	7
N	12	21	37	63	102	155	216	275

t	8	9	10	11	12	13	14	15
N	323	358	381	394	402	406	409	410


Number of Bison
450
400
350
300
250
200
150
100
50
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
Time in Years

The Logistic Model

The function model used for generating the population numbers for bison in the wildlife reserve is

$$N(t) = \frac{12.36}{0.03 + e^{-0.6t}}$$

The Law of Natural Growth and Decay

$$\frac{dN}{dt} = kN$$

General Solution: $N(t) = N_0 e^{kt}$

The Logistic Model

$$\frac{dN}{dt} = kN - aN^2$$

General Solution: $N(t) = \frac{LC_0 e^{kt}}{1 + C_0 e^{kt}}$

When ecologists attempt to construct a mathematical function for the population growth according to the Logistic Model they typically examine the ratio of $\frac{\Delta N}{t}$ to N, i.e., $\frac{\Delta N}{N}$ (note that $\Delta t = 1$ in this situation).

The theoretical relationship between $\frac{\Delta N}{t}$ and N is given by $\frac{\Delta N}{N} = k - aN$. Notice that $\frac{\Delta N}{N}$ is a linear function in terms of N. Thus, if the data of $\frac{\Delta N}{t}$ versus N is *almost* linear in terms of N, then the data suggests that the Logistic Model is a good model for predicting the data.