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Applications of the Definite Integral

Summary

The new Advanced Placement Course Description now includes more general, conceptual applications of the definite integral. The usual geometric applications (area, volume, arc length) are still important and required, but other problems involve the integral as an accumulation function. The purpose of this presentation is to discuss some unusual, engaging geometric applications as well as different applications that require the definite integral.

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Introduction

Definite integrals aren't just for area any more!

1. *Any* definite integral may be interpreted as a signed area.
2. Area, volume, arc length, work, mass, fluid pressure, and accumulated financial value are quantities that may be calculated with definite integrals.
3. The most important components of these problems are *constructing* the correct integral and *interpreting* the results.

Two views of the definite integral: When using the definite integral to solve various problems, it is useful to consider two different interpretations.

1. A limit of approximating sums: The definite integral is formally defined as a limit of approximating sums. Using right sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

   The integral *adds up* small parts, each of the form $f(x_i)\,\Delta x_i$.

2. Accumulated change in an antiderivative: The Fundamental Theorem of Calculus states

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

   where F is any antiderivative of f on $[a, b]$.

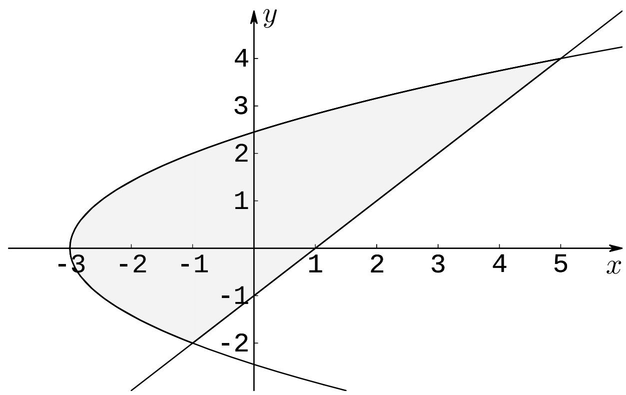
   The difference $F(b) - F(a)$ represents the accumulated change (or net change) in F over the interval $[a, b]$.

   To find the accumulated change in F over $[a, b]$, integrate f, the rate function associated with F, over the interval $[a, b]$.

Which view is better: sum or antiderivative?

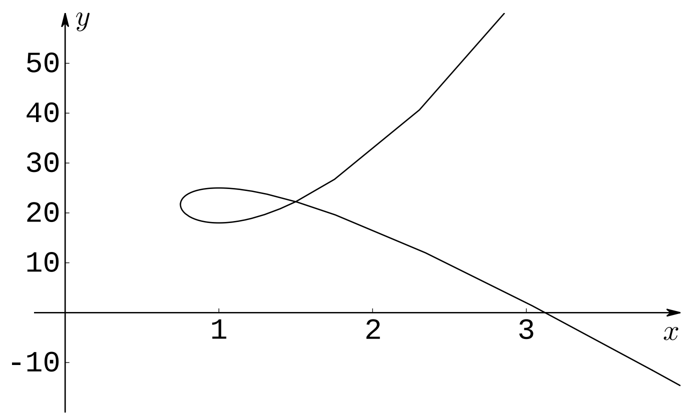
1. Often we need to decide which view (or interpretation) of the definite integral is the correct one for a given application. It could be that an approximating sum is acceptable or that a precise symbolic antiderivative is more appropriate.
2. If an integral is presented in symbolic form, then antidifferentiation seems reasonable.
3. For data given graphically or in a table, approximating sums are the logical choice.

1. (Stewart) Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

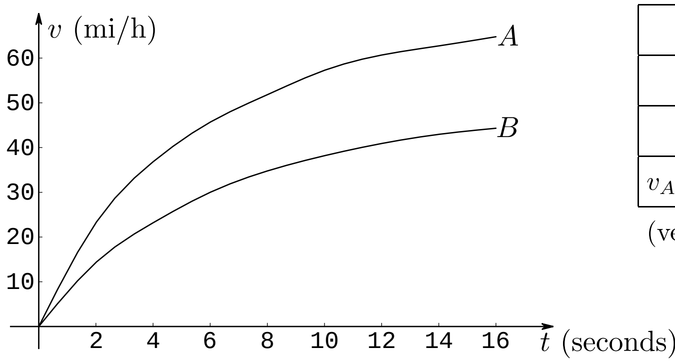


2. (Stewart) Estimate the area enclosed by the loop of the curve with parametric equations

$$x = t^2 + t + 1 \qquad y = 3t^4 - 8t^3 - 18t^2 + 25.$$



3. (Stewart) The graph below shows the velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between these two curves represent? Use Simpson's Rule to estimate this area.



t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
$v_A - V_B$	0	13	20	23	25	28	29	29	30

(velocity in f/s)

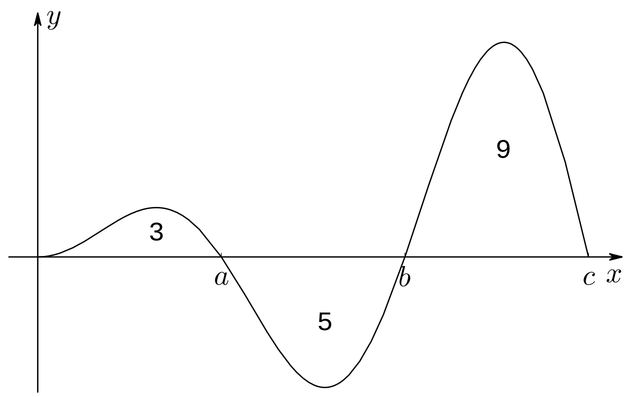
4. (FDWK) The graph below shows the velocity in cm/sec of a particle moving along the x axis. The numbers in the graph are the areas of the enclosed regions.

   (a) What is the displacement of the particle between $t = 0$ and $t = c$?

   (b) What is the total distance traveled by the particle in the same time period?

   (c) Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, b]$?

   (d) Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, c]$?



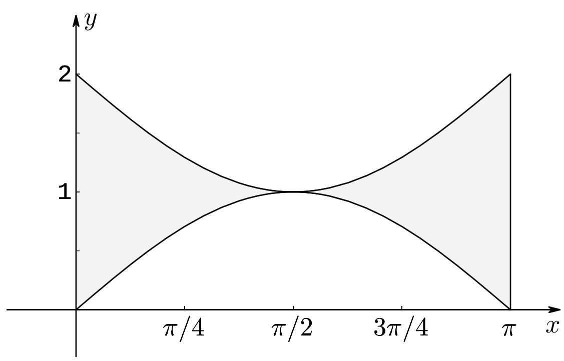
5. (FDWK) For each positive integer k, let A_k denote the area of the butterfly–shaped region enclosed between the graphs of $y = k\sin kx$ and $y = 2k - k\sin kx$ on the interval $[0, \pi/k]$. The regions for $k = 1$ and $k = 2$ are shown in the figure below.

(a) Find the area of the two figures

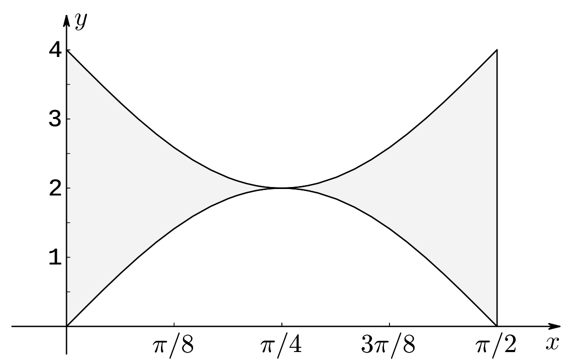
(b) Set up a definite integral for A_k. Find A_1.

(c) Make a conjecture about the area A_k, $k \geq 3$.

(d) What is $\lim_{k\to\infty} A_k$?



$k = 1$



$k = 2$

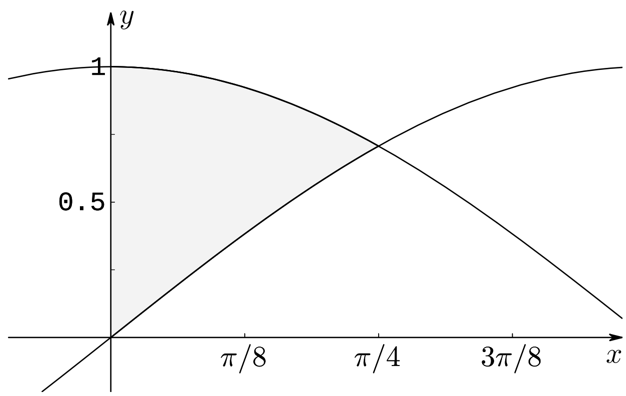
6. (FDWK) Population density measures the number of people per square mile inhabiting a given living area. The population density of a certain city decreases as you move away from the center of the city. The density at a distance r miles from the city center can be approximated by the function $10000(2 - r)$.
   (a) If the population density tails off to zero at the edges of the city, what is the city radius?
   (b) A thin ring of area around the center of the city has thickness Δr and radius r. If the ring is straightened out, it becomes a rectangular strip. What is the area of this region?
   (c) Explain why the population of the ring in part (b) is given by $10000(2 - r)(2\pi r)\,\Delta r$.
   (d) Find the total population of the city by setting up and evaluating a definite integral.

7. (FDWK) Let R be the shaded region in the first quadrant enclosed by the y axis and the graphs of $y = \sin x$ and $y = \cos x$, as shown in the figure below.

   (a) Find the area of the region R.

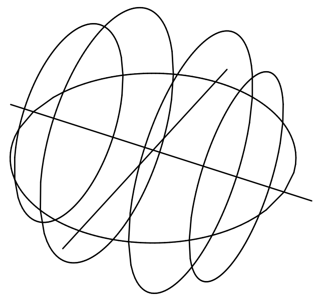
   (b) Find the volume of the solid when R is revolved about the x axis.

   (c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x axis are squares.

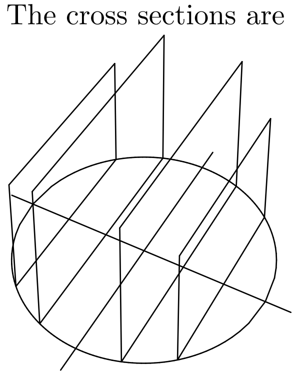


8. (FDWK) Find a formula for the area $A(x)$ of the cross sections of the solid perpendicular to the x axis. Find the volume of the resulting solid. The solid lies between planes perpendicular to the x axis at $x = -1$ and $x = 1$. In each case, the cross sections perpendicular to the x axis between these planes run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

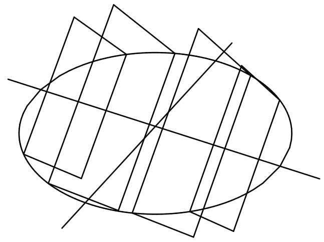
(a) The cross sections are circular disks with diameters in the xy plane.



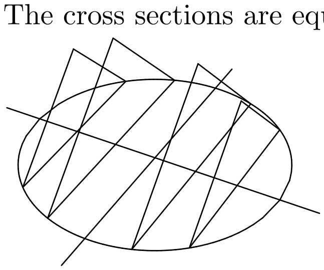
(b) The cross sections are squares with bases in the xy plane.



(c) The cross sections are squares with diagonals in the xy plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)



(d) The cross sections are equilateral triangles with bases in the xy plane.



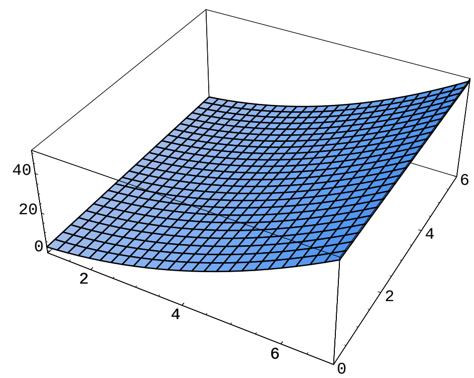
9. (ETS) A 3–dimensional object has square base, defined by $1 \leq x \leq 7$ and $0 \leq y \leq 6$. The height of the object at any point (x, y) in the base is given by $h(x) = x^2 + 1$.

   (a) Approximate the volume of this object by dividing the base into 3 strips of width 2 and using the height at the left edge of each strip.

   (b) Write a Riemann sum that approximates the volume of the object, dividing the base into n strips of width $\Delta x = \dfrac{7-1}{n}$ and using the height at the left edge of each strip.

   (c) Write an integral for the volume of the object.

   (d) Find the volume of the object.



10. (Consumer Surplus: Stewart) A typical demand curve is shown in the figure below; a graph of $p = p(x)$. Suppose X is the amount of the commodity currently available: $P = p(x)$ is the current selling price.

   (a) Divide $[0, X]$ into n subintervals, each of length $\Delta x = X/n$.
   Let $x_i^* = x_i$ be the right endpoint of the ith subinterval.

   (b) If, after the first x_{i-1} units were sold, a total of only x_i units had been available and the price per unit had been set at $p(x_i)$ dollars, then the additional Δx units could have been sold.

   (c) The consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them.

   (d) In paying only P dollars they have saved an amount of

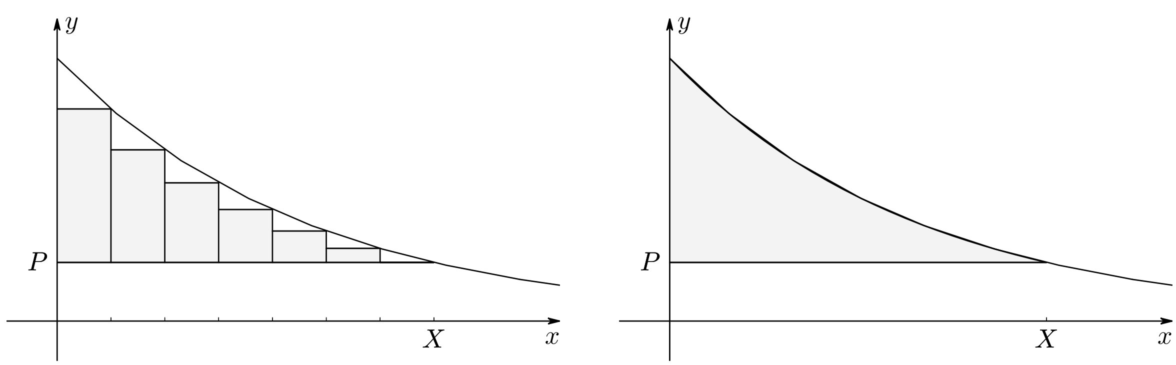
   $$\text{(savings per unit)(number of units)} = [p(x_i) - P]\,\Delta x.$$

   (e) Considering similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

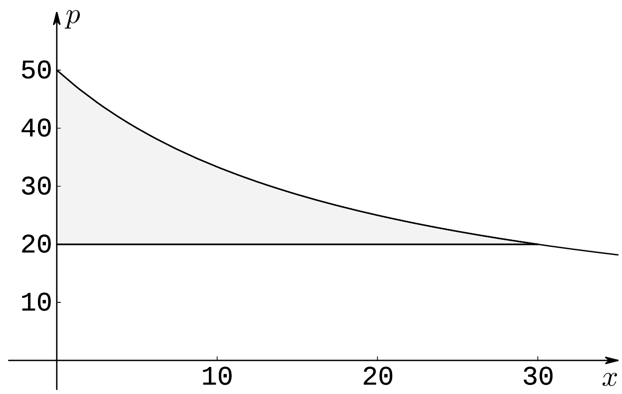
   $$\sum_{i=1}^{n} [p(x_i) - P]\,\Delta x.$$

   (f) As $n \to \infty$ this Riemann sum approaches the integral

   $$\int_0^X [p(x) - P]\,dx.$$



11. A demand curve is given by $p = 1000/(x + 20)$. Find the consumer surplus when the selling price is \$20.



12. (Blood Flow: Stewart) When we consider the flow of blood through a blood vessel, such as a vein or an artery, we may take the shape of the blood vessel to be a cylindrical tube with radius R and length l. Because of friction at the walls of an artery or vein, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the law of laminar flow (Poisseuille):

$$v = \frac{P}{4\eta l}\left(R^2 - r^2\right)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

In order to compute the flux (volume per unit time), we consider smaller, equally spaced radii $r_1, r_2, \ldots$. The approximate area of the annulus with inner radius r_{i-1} and outer radius r_i is

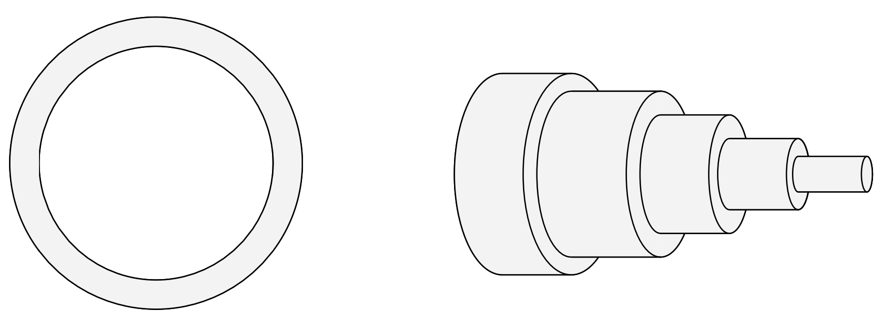
$$2\pi r_i \,\Delta r \quad \text{where} \quad \Delta r = r_i - r_{i-1}.$$

If Δr is small, then the velocity is almost constant throughout this annulus and can be approximated by $v(r_i)$. So, the volume of blood per unit time that flows across the annulus is approximately

$$(2\pi r_i \,\Delta r) v(r_i) = 2\pi r_i v(r_i)\,\Delta r$$

and the total volume of blood that flows across a cross–section per unit time is approximately

$$\sum_{i=1}^{n} 2\pi r_i v(r_i)\,\Delta r.$$



The velocity and the volume per unit time increase toward the center of the blood vessel. The approximation gets better as n increases. Take the limit to obtain the exact value of the flux (or discharge), which is the volume of blood that passes a cross–section per unit time:

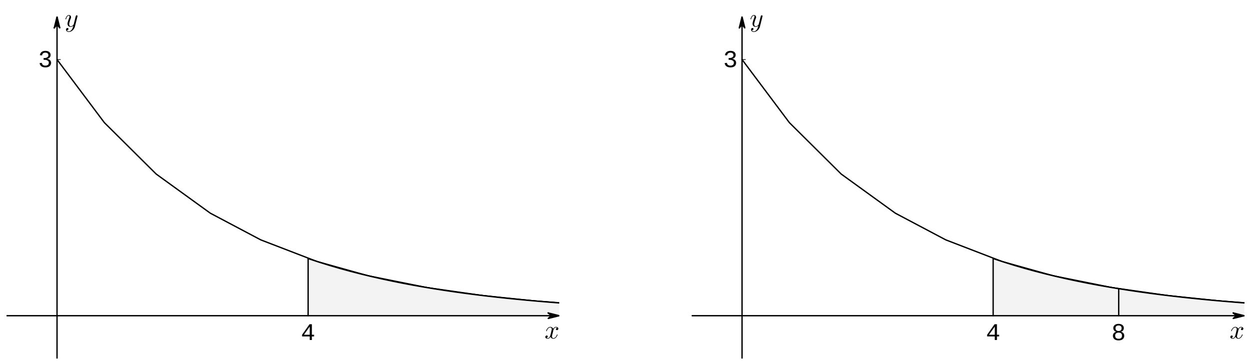
13. The probability distribution for a continuous random variable X is given by a probability density function (pdf) such that the probability that X takes on values between a and b is the area under the curve between a and b.

The failure time for a certain computer component is modeled by an exponential random variable with pdf given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3e^{-3x} & \text{if } x \geq 0 \end{cases}$$

(a) Find the probability the part lasts for at least 4 years.

(b) Suppose the part lasts for 4 years, find the probability it lasts for at least another 4 years.



14. (Stewart) A solid is generated by rotating about the x axis the region bounded by the x axis, the y axis, and the curve $y = f(x)$, where f is a positive function and $x \geq 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for all $b > 0$. Find the function f.

15. (Stewart) A high–tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate of $g = g(t)$, where t is measured in months. The company wants to determine the optimum time to replace the system.

   (a) Let

$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$$

   Find the critical number of C.

   (b) Suppose that

$$f(t) = \begin{cases} \dfrac{V}{15} - \dfrac{V}{450}t & 0 < t \leq 30 \\ 0 & t > 30 \end{cases} \qquad \text{and} \quad g(t) = \frac{Vt^2}{12900}, \quad t > 0$$

   Determine the length of time T for the total depreciation $D(t) = \displaystyle\int_0^t f(s)\,ds$ to equal the initial value V.

   (c) Determine the absolute minimum of C on $(0, T]$.

   (d) Sketch the graphs of C and $f + g$ on the same coordinate system. Verify the result in part (a).

16. (Hughes–Hallett) The density of oil in a circular oil slick on the surface of the ocean at a distance r meters from the center of the slick is given by $\rho(r) = 50/(1+r)$ kg/m^2.

   (a) If the slick extends from $r = 0$ to $r = 10000$ m, find a Riemann sum approximating the total mass of oil in the slick.

   (b) Find the exact value of the mass of oil in the slick.

   (c) Within what distance r is half the oil of the slick contained?

17. (Stewart) The hydrogen atom is composed of one proton in the nucleus and one electron, which moves about the nucleus. In the quantum theory of atomic structure, it is assumed that the electron does not move in a well–defined orbit. Instead, it occupies a state known as an *orbital*, which may be thought of as a *cloud* of negative charge surrounding the nucleus. At the state of lowest energy, called the *ground state*, or 1*s orbital*, the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability density function

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \qquad r \geq 0$$

where a_0 is the *Bohr radius* ($a_0 \approx 5.59 \times 10^{-11}$ m). The integral

$$P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$$

gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.

(a) Sketch the graph of the probability density function p.

(b) Determine the probability that the electron will be within the sphere of radius $4a_0$ centered at the nucleus.

(c) The most probable distance (expected value) of the electron from the nucleus in the ground state of the hydrogen atom is given by

$$E = \int_0^\infty r p(r)\, dr$$

Find the value of E.