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The Accumulation Function

Summary

The Fundamental Theorem of calculus gives the precise inverse relationship between the derivative and the integral. The statement of the theorem includes the definition of an *area so far*, or *accumulation* function. This presentation focuses on the accumulation function, some typical applications, and some other challenging AP type problems.

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Introduction:

The first part of the Fundamental Theorem of Calculus deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)\,dt$$

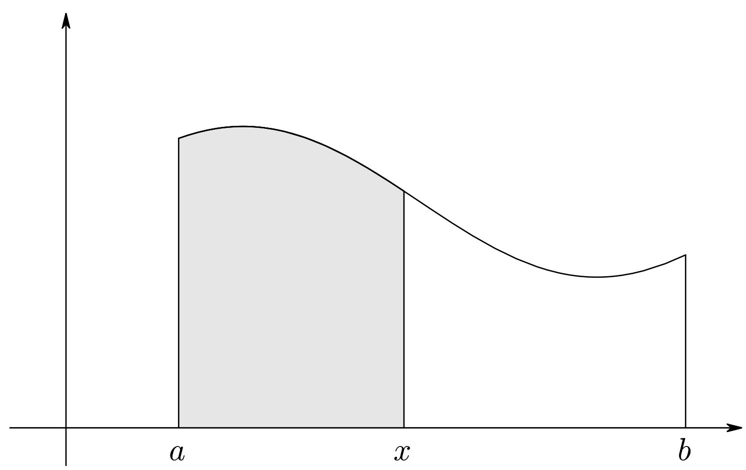
where f is a continuous function on $[a, b]$ and x varies between a and b.

1. The function g depends only on x, which is the upper limit in the integral.

2. If x is a fixed number, then the integral is a definite number.

3. If x varies, then the integral varies and defines a function of x denoted by $g(x)$.

If f is a positive function, then $g(x)$ may be interpreted as the area under the graph of f from a to x, where x may vary from a to b. g is an *area so far*, or accumulation, function.

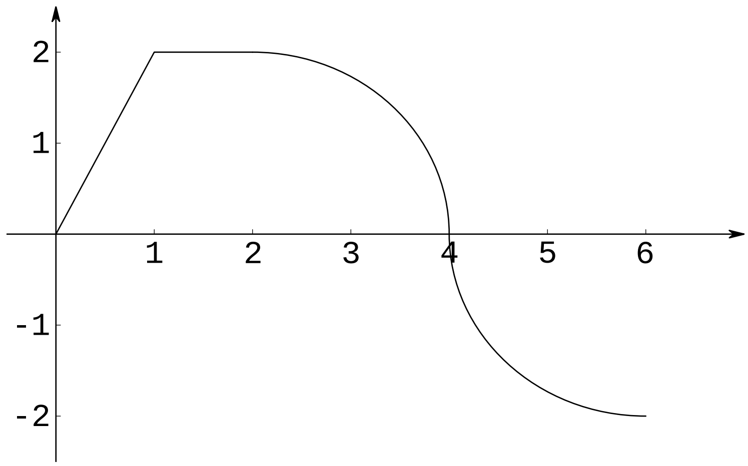


Note: If f is any continuous function on $[a, b]$ (positive and negative), then the function g is a *net* area so far function.

Example 1: Suppose f is the function whose graph is given below, and define $g(x) = \int_0^x f(t)\,dt$.

(a) Find the values of $g(0), g(1), g(2), g(4), g(6)$.

(b) Sketch a rough graph of g.



Example 2: Suppose $g(x) = \int_1^x (t^2 - 1)\, dt$. Find a formula for $g(x)$ and calculate $g'(x)$.

This suggests the FTC, Part 1:

The Fundamental Theorem of Calculus, Part 1:

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\, dt \quad a \leq x \leq b$$

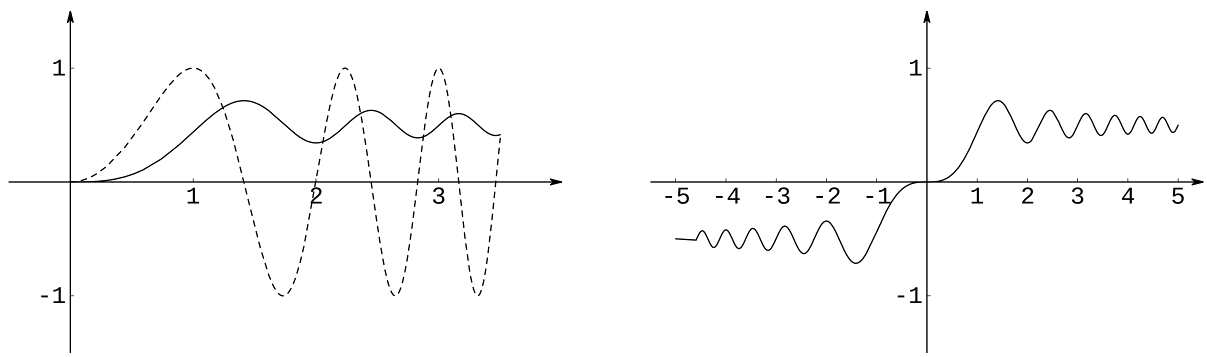
is an antiderivative of f, that is, $g'(x) = f(x)$ for $a < x < b$.

Example 3: Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^3}\, dt$.

Example 4: Find the derivative of the function $g(x) = \int_0^{x^2} \sqrt{1 + t^2}\, dt$.

Example 5: Consider the *Fresnel function* defined by $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$.

The figures below show the graphs of $y = f(x) = \sin(\pi x^2/2)$ (dashed line) and $y = S(x)$ (solid line).



(a) At what values of x does S have local maximum values?

(b) On what intervals is the function concave upward?

(c) Solve the following equation correct to two decimal places: $\int_0^x \sin(\pi t^2/2)\,dt = 0.2$.

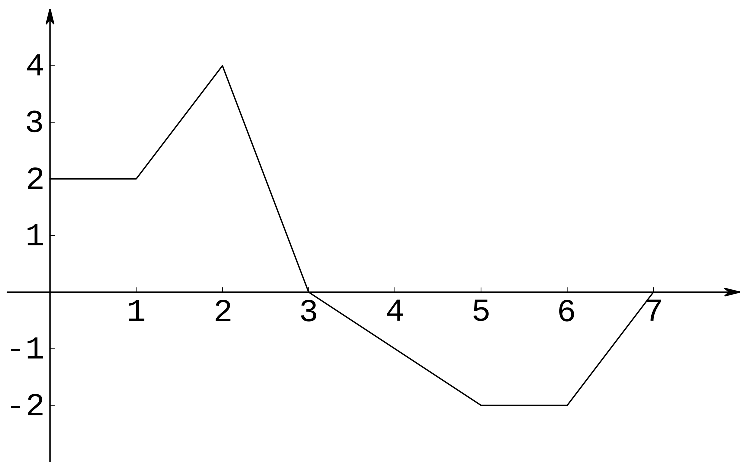
Example 6: Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(0), g(1), g(2), g(3)$, and $g(6)$.

(b) On what interval is g increasing?

(c) Where does g have a maximum value?

(d) Sketch a rough graph of g.



Example 7: Find the derivative of each function.

(a) $F(x) = \int_x^{10} \tan\theta \, d\theta$

(b) $h(x) = \int_2^{1/x} \tan^{-1} t \, dt$

(c) $g(x) = \int_{e^x}^{0} \sin^3 t \, dt$

(d) $j(x) = \int_{2x}^{3x} \frac{t^2 - 1}{t^2 + 1} \, dt$

(e) $k(x) = \int_{\cos x}^{5x} \cos(t^2) \, dt$

Example 8: Suppose $F(x) = \int_1^x f(t)\,dt$, where $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du$. Find $F''(2)$.

Example 9: Suppose $g(x) = \int_0^x \frac{1}{1+t+t^2}\,dt$. Find the intervals on which the graph of $y = g(x)$ is concave upward.

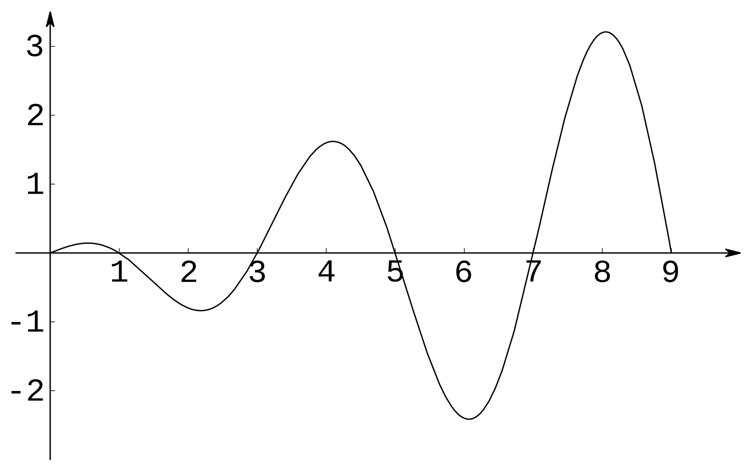
Example 10: Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) At what values of x do the local maximum and minimum values of g occur?

(b) Where does g attain its absolute maximum value?

(c) On what intervals is g concave downward?

(d) Sketch the graph of g.



Example 11: Consider the sine integral function $\mathrm{Si}(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$.

The integrand is not defined at $t = 0$ but the limit is 1. Define $f(0) = 1$ to make f continuous everywhere.

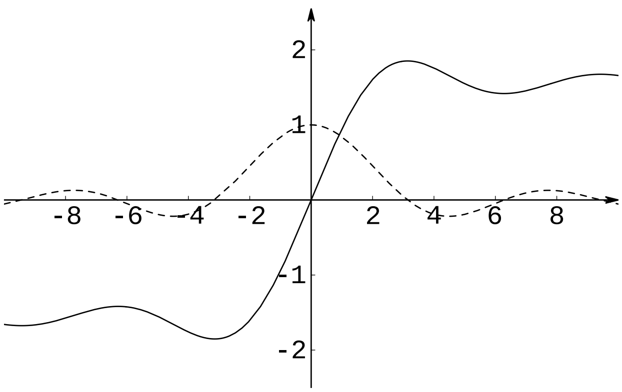
(a) Carefully sketch the graph of $y = \mathrm{Si}(x)$.

(b) At what values of x does this function have local maximum values?

(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does this function have horizontal asymptotes?

(e) Solve the following equation correct to one decimal place: $\displaystyle\int_0^x \frac{\sin t}{t}\,dt = 1$.



Example 12: Find a function f and a number a such that

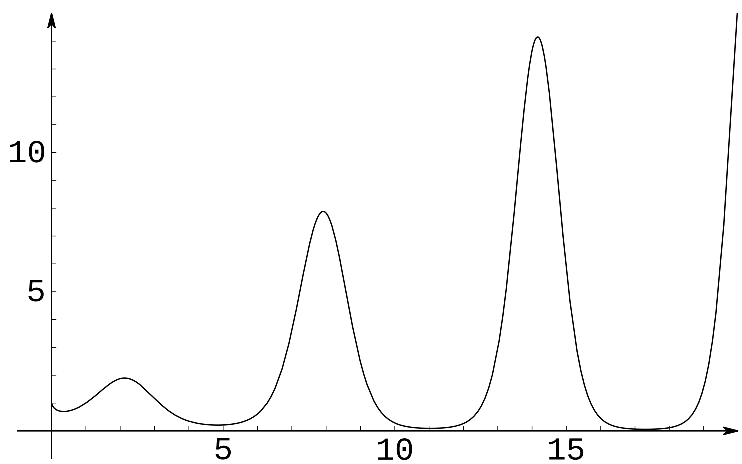
$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$$

for all $x > 0$.

Example 13: Let $f(x) = x^{\sin x}$.

(a) Carefully sketch a graph of $y = f(x)$ for $0 < x \leq 20$.

(b) Using the answer to part (a), carefully sketch a graph of $g(x) = \int_0^x f(t)\,dt$.



Example 14: Suppose $g(x) = \operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$ (the *error function*).

(a) Find the derivative of $x \operatorname{erf}(x)$.

(b) Find the derivative of $\operatorname{erf}(\sqrt{x})$.

(c) Find an expression for $F(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^x e^{-t^2/2}\, dt$ in terms of $\operatorname{erf}(x)$.

(d) Find an expression for $\sqrt{\dfrac{2}{\pi}} \displaystyle\int_{x_1}^{x_2} e^{-t^2/2}\, dt$ in terms of $\operatorname{erf}(x)$.

Example 15: Suppose f is the differentiable function shown in the accompanying graph and that the position at time t (seconds) of a particle moving along a coordinate axis is $s(t) = \int_0^t f(x)\,dx$ meters. Use the graph to answer the following questions.

(a) What is the particle's velocity at time $t = 5$?

(b) Is the acceleration of the particle at time $t = 5$ positive or negative?

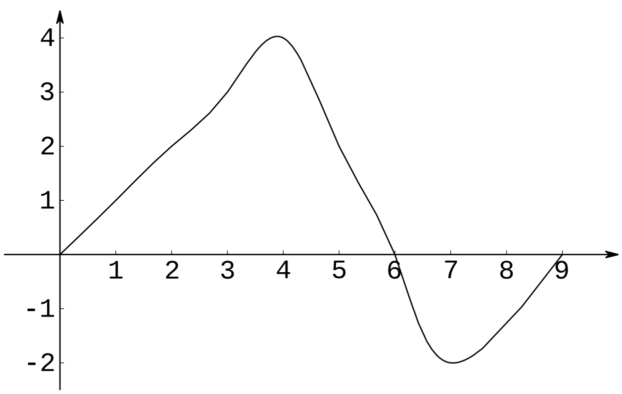
(c) What is the particle's position at time $t = 3$?

(d) At what time during the first 9 seconds does s have its largest value?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? Away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?



Example 16: A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate of $g = g(t)$, where t is measured in months. The company wants to determine the optimum time to replace the system.

(a) Let

$$C(t) = \frac{1}{t} \int_0^t [f(s) + g(s)]\, ds$$

Show that the critical numbers of C occur at the numbers t where $C(t) = f(t) + g(t)$.

(b) Suppose that

$$f(t) = \begin{cases} \dfrac{V}{15} - \dfrac{V}{450} t & 0 < t \leq 30 \\ 0 & t > 30 \end{cases} \qquad \text{and} \quad g(t) = \frac{Vt^2}{12900}, \quad t > 0$$

Determine the length of time T for the total depreciation $D(t) = \displaystyle\int_0^t f(s)\, ds$ to equal the initial value V.

(c) Determine the absolute minimum of C on $(0, T]$.

(d) Sketch the graphs of C and $f + g$ on the same coordinate system. Verify the result in part (a).