Proof of Crossword Puzzle Record

Kevin K. Ferland
kferland@bloomu.edu
Bloomsburg University, Bloomsburg, PA 17815

Abstract

We prove that the maximum number of clues possible for a 15×15 daily New York Times crossword puzzle is 96 and and determine all possible puzzle grids with 96 clues.

1 Introduction.

Daily New York Times crossword puzzles are constructed in a 15×15 grid, each of whose squares is colored black or white. Each maximal linear segment of white squares is then attached to a *clue* for the puzzle. The grid itself must be constructed according to certain *structure rules*.

1. *Connectivity*: The centers of any two white squares in the grid can be joined by a path consisting of horizontal and vertical line segments that meet in the center of and only pass through white squares.

2. *Symmetry*: The grid looks the same if rotated 180 degrees.

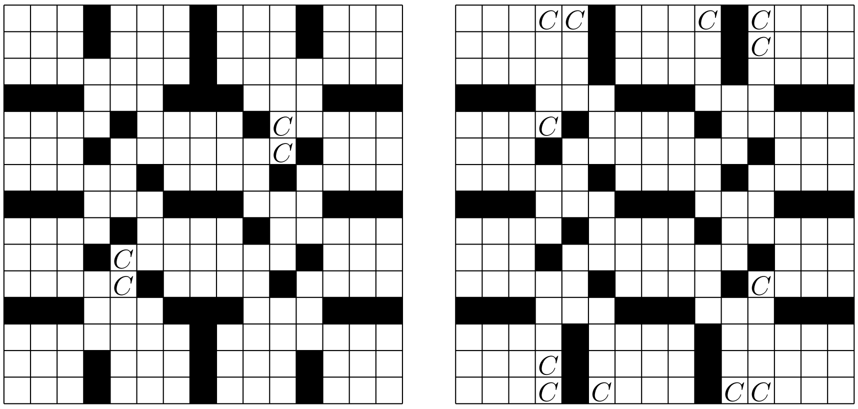
3. *Three+*: Each clue's answer must be at least 3 characters long.

Two examples of possible puzzle grids are pictured in Figure 1, if we ignore the C's included there. Those C's mark *cheater squares*, which may be switched to black or white, while following the structure rules, without changing the number of clues in the puzzle. The following result is proven in Section 3.

Theorem 1.1. *The maximum number of clues possible for a 15×15 crossword grid satisfying Connectivity, Symmetry, and Three+ is 96. Moreover, up to horizontal reflection, vertical reflection, and the inclusion of cheater squares, all possible grids requiring 96 clues are pictured in Figure 1, where cheater squares have been marked with a C.*

2 Fundamental Grid Structure.

Starting with 1, we number the rows of a grid from top to bottom and the columns from left to right. Thus, position (i, j) in the grid is the square in row



a *b*

Figure 1: All Crosswords with 96 Clues

i and column j. We refer to row 8 as the *center row*, and column 8 as the *center column*. Rows 1 through 7 are the *top rows*, rows 8 through 15 are the *bottom rows*, columns 1 through 7 are the *left columns*, and columns 8 through 15 are the *right columns*. We denote the number of clues in row i by r_i and the number in column i by c_i. By considering the cases in which the center square $(8, 8)$ is black or white, we get the following immediate consequence of the Symmetry rule.

Lemma 2.1. *r_8, the number of clues in row 8, and c_8, the number of clues in column 8 have the same parity*

We let Across be the total number of row clues, Down the number of column clues, and Clues the total number of clues. Of course, Clues = Across + Down. As a consequence of the Symmetry rule, we make frequent use of the equations

$$\begin{aligned} &\text{Across} = r_8 + 2[r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7] \text{ and} \\ &\text{Down} = c_8 + 2[c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7]. \end{aligned} \tag{2.1}$$

That is, we can focus our counting on the center, top, and left portions of the grid. Moreover, we get the following immediate consequence of Lemma 2.1

Corollary 2.2. *The number of clues in any 15×15 crossword grid satisfying the structure rules is always even.*

As a consequence of the Three+ rule, there can be at most 4 clues in a row or column. A FOUR is a row or column that requires exactly 4 clues. In Figure 1a, the first column is a FOUR, and we note that this is the unique way to obtain a FOUR. Similarly, a ONE, a TWO, or a THREE is a row or column that requires exactly 1, 2, or 3 clues, respectively. Each of those can be obtained in multiple ways. For example, both the third and fourth rows of Figure 1a are TWO's.

3 The Proof of Theorem 1.1.

By Symmetry, we may assume that there are at least as many off-center FOUR's in the columns as the rows. We consider several cases, based on the placement of column FOUR's in the left-hand side of the grid. By Connectivity and Three+, we cannot have 7 left FOUR's.

3.1 5 or 6 left FOUR's

By Connectivity and Three+, the left FOUR's must be left-justified. It follows that $r_4 = r_8 = 1$. By Connectivity and Three+, it is impossible to place black squares in row 3 to make $r_3 \geq 3$. Hence $r_3 \leq 2$. By (2.1), we now have Down $\leq 4 + 2[6(4) + 3] = 58$ and Across $\leq 1 + 2[5(3) + 1 + 2] = 37$. Thus, Clues < 96.

3.2 4 left FOUR's

By Connectivity and Three+, the left FOUR's can only be placed in certain ways. Based on the placement of the FOUR's, we refer to possible cases distinguished by the black squares shown in Figure 2.

The placement in Figure 2a. We see that $r_4 \leq 2$, $r_8 \leq 2$, and FOUR's are not possible in any of the rows. By (2.1), Down $\leq 4 + 2[4(4) + 3(3)] = 54$ and Across $\leq 2 + 2[2 + 6(3)] = 42$. Hence, 96 clues is only possible if $r_3 = 3$. That then forces the squares marked with a T in row 3 to be black, and the remaining T's must also be black by Symmetry and Three+. However, it is then not possible to have $c_6 = 3$. Thus, Clues < 96.

The placement in Figure 2b. By Three+, the squares marked with an X must be black. Also, the squares marked with a U are either all black or all white. The same is true for those marked with an L. Since column 8 cannot be entirely black, we have three cases to consider. In each case, $r_4 \leq 2$ and $r_8 \leq 2$.

Case 1: The U's are white, and the L's are black. We have $c_8 = 2$. Since there is no square where column 6 can be cut into multiple clues, we have $c_6 = 1$. In this case, the first three rows cannot be FOUR's, and rows 4, 5, and 6 cannot all be FOUR's. Hence $r_4+r_5+r_6 \leq 11$. By (2.1), Down $\leq 2+2[4(4)+2(3)+1] = 48$ and Across $\leq 2 + 2[2 + 3(3) + 11] = 46$. Thus, Clues ≤ 94.

Case 2: The U's are black, and the L's are white. By (2.1), we have Down $\leq 2+2[4(4)+3(3)] = 52$. In rows 5, 6, and 7, there cannot be any FOUR's. So $r_5, r_6, r_7 \leq 3$. In row 3, the structure rules do not allow any square except for $(3, 8)$ to be black. Therefore, $r_3 = 2$, and Across $\leq 2 + 2[2(4) + 2(2) + 3(3)] = 44$. Thus, Clues ≤ 96, and equality is possible only with all equalities above. Hence, rows 1 and 2 are FOUR's, and columns 4 and 12 can then only be THREE's in one way. Now column 6 can only be a THREE in two ways, which are horizontal reflections of each other. That then forces the structure of column 5, and the grids represented by Figure 1a are obtained.

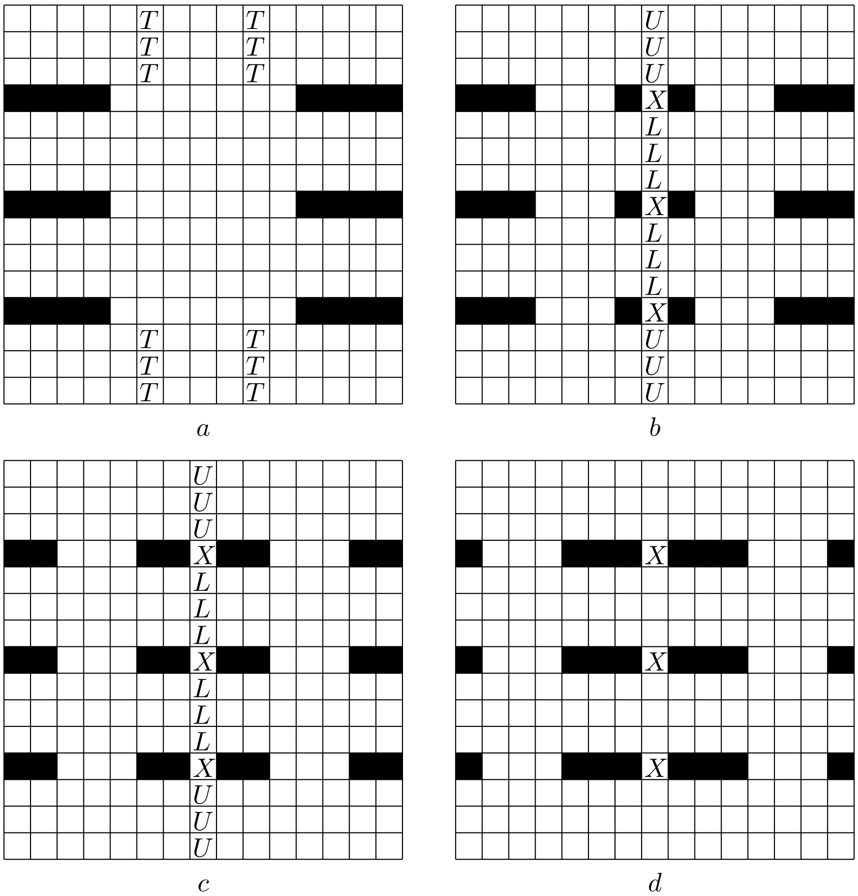


Figure 2: 4 left FOUR's

Case 3: The U's and L's are all white. Here $c_8 = 4$, and none of the rows can be FOUR's. By (2.1), Down $\leq 4 + 2[4(4) + 3(3)] = 54$ and Across $\leq 2 + 2[2 + 6(3)] = 42$. Therefore Clues ≤ 96, and equality is only possible with all equalities above. By Three+, the columns with the black squares used to make row 3 into a THREE must be the same for rows 1 and 2. Since columns 5 and 11 are THREE's, it follows from Symmetry that we cannot have both $(3, 5)$ and $(3, 11)$ black together. Hence, the black squares in row 3 must be chosen like those in Figure 1b or its vertical reflection. That forces the structure of the remaining columns, and the grids represented by Figure 1b are obtained.

The placement in Figure 2c. The squares marked with an X must be black, and those marked with a U as well as those marked with an L are either all black or all white. Since $c_8 \neq 0$, we have three cases to consider. In each case, $r_4 \leq 2$ and $r_8 \leq 2$. Also, by Three+, $c_3 = 1$.

Case 1: The U's are white, and the L's are black. We have $c_8 = 2$. By Three+, no squares in column 5 can be black. So $c_5 = 1$. In this case, the first 3 rows cannot be FOUR's. By (2.1), Down $\leq 2 + 2[4(4) + 2(1) + 3] = 44$ and Across $\leq 2 + 2[2 + 3(3) + 3(4)] = 48$. Thus, Clues ≤ 92.

Case 2: The U's are black, and the L's are white. We have $c_8 = 2$. In this case, rows 5, 6, and 7 cannot be FOUR's. By (2.1), Down $\leq 2 + 2[4(4) + 1 + 2(3)] = 48$ and Across $\leq 2 + 2[2 + 3(4) + 3(3)] = 48$. Hence 96 clues is possible only if $r_1 = r_2 = r_3 = 4$. However, it then is not possible to have $c_4 = 3$. Thus, Clues < 96.

Case 3: The U's and L's are all white. Here $c_8 = 4$, and none of the rows can be FOUR's. By (2.1), Down $\leq 4 + 2[4(4) + 1 + 2(3)] = 50$ and Across $\leq 2 + 2[2 + 6(3)] = 42$. Thus, Clues ≤ 92.

The placement in Figure 2d. The squares marked with an X must be black, and $r_4, r_8 \leq 2$. By Three+, $c_2 = c_3 = 1$. By Connectivity, we cannot have all FOUR's in rows 1, 2, and 3. The same is true in rows 5, 6, and 7. Moreover, by Three+, $r_1 + r_2 + r_3 \leq 10$ and $r_5 + r_6 + r_7 \leq 10$. By (2.1), Down $\leq 4 + 2[4(4) + 2(1) + 3] = 46$ and Across $\leq 2 + 2[2 + 10 + 10] = 46$. Thus, Clues ≤ 92.

The placement in columns 4, 5, 6, and 7. In this case, $c_4 = c_5 = c_6 = c_7 = 4$, and $(4, 8)$, $(8, 8)$, and $(12, 8)$ must be black. By Connectivity and Three+, we must have each $r_i \leq 2$, and hence Across $\leq 15(2) = 30$. By (2.1), Down $\leq 4 + 2[4(4) + 3(3)] = 54$, and thus, Clues ≤ 84.

3.3 3 left FOUR's

Based on the placement of the FOUR's, we refer to possible cases distinguished by the black squares shown in Figure 3.

The placement in Figure 3a. Observe that $r_4, r_8 \leq 2$, and $r_3 \neq 4$ by Connectivity and Three+.

Claim 1: $c_8 \neq 4$. Otherwise, no $r_i = 4$, since FOUR's are unique. By (2.1), Across $\leq 2 + 2[2 + 6(3)] = 42$ and Down $\leq 4 + 2[3(4) + 4(3)] = 52$. Thus, Clues ≤ 94.

Claim 2: $r_7 \neq 4$. Otherwise, by Symmetry, $r_9 = 4$, and by Three+, $(8, 4)$, $(8, 8)$, and $(8, 12)$ are black. Note that $r_5 \neq 4$ by Connectivity and Three+. Also, $c_4, c_8 \leq 2$, since there are few places for black squares in columns 4 and 8. By (2.1), Down $\leq 2 + 2[3(4) + 2 + 3(3)] = 48$. We need only consider two top row FOUR's in addition to row 7. If $r_1 = r_2 = 4$, then $r_3 \leq 2$ and Across $\leq 2 + 2[3(4) + 2(2) + 2(3)] = 46$. If $r_1 = r_6 = 4$, then $r_5 \leq 2$ and similarly Across ≤ 46. Thus, in both possible cases, Clues ≤ 94.

Claim 3: $r_2 \neq 4$. Otherwise, $r_1 = 4$. By Three+, $r_3 \leq 2$ and $r_5 \neq 4$. If $r_6 = 4$, then Three+ forces $r_3 = 1$. Hence, in all cases $r_3 + r_5 \leq 5$. By (2.1),

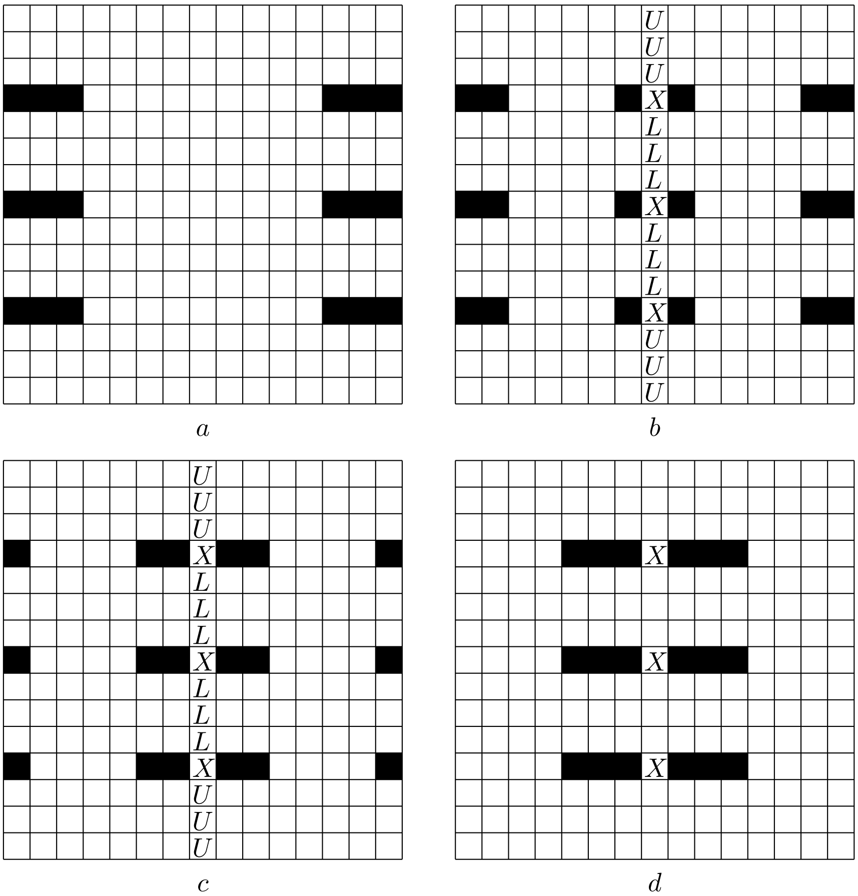


Figure 3: 3 left FOUR's

Across $\leq 2 + 2[2(4) + 5 + 2(3) + 2] = 44$ and Down $\leq 3 + 2[3(4) + 4(3)] = 51$. Thus, Clues < 96.

This leaves a single possibility to consider for three top row FOUR's, namely $r_1 = r_5 = r_6 = 4$. By Three+, there can be no black squares in row 2. So $r_2 = 1$. By (2.1), Across $\leq 2+2[3(4)+1+2+2(3)] = 44$ and Down $\leq 3+2[3(4)+4(3)] = 51$. Thus, Clues < 96.

The placement in Figure 3b. The squares marked with an X must be black, and $r_4, r_8 \leq 2$. By (2.1), Down $\leq c_8 + 2[3(4) + c_4 + 3(3)] = 42 + 2c_4 + c_8$. Of course, $c_4 \leq 3$. We consider three cases based on the squares marked U and L.

Case 1: The U's are white, and the L's are black. We have $c_8 = 2$ and $r_1, r_2, r_3 \neq 4$. If $r_7 = 4$, then we also have $r_9 = 4$, by Symmetry. Moreover, Three+ forces all of row 8 to be black. Hence $r_7 \neq 4$. By (2.1), Across $\leq$

$2+2[2+2(4)+4(3)] = 46$. Therefore Clues ≤ 96, and equality is only possible with all equalities above. However, having $r_5 = r_6 = 4$ makes it impossible to have column 5 be a THREE. Thus, Clues < 96.

Case 2: The U's are black, and the L's are white. We have $c_8 = 2$ and $r_5, r_6, r_7 \neq 4$. By (2.1), Across $\leq 2+2[2+2(4)+r_3+3(3)] = 40+2r_3$. Thus, Clues $\leq 84+2r_3+2c_4$. If $r_3 = 4$, then column 4 is forced to be a TWO. Hence, whether $r_3 = 4$ or $r_3 \leq 3$, we have Clues ≤ 96. In both cases, equality is only possible with all corresponding equalities above. If $r_3 = 4$, then column 3 can only be a THREE in ways which would violate Connectivity. If $r_3 = 3$, then this together with the FOUR's in rows 1 and 2 make it impossible for column 4 to be a THREE. Thus, Clues < 96.

Case 3: The U's and L's are all white. Here $c_8 = 4$, and none of the rows can be FOUR's. By (2.1), Across $\leq 2+2[2+6(3)] = 42$. Thus, Clues ≤ 94.

The placement in Figure 3c. The squares marked with an X must be black, and $r_4, r_8 \leq 2$. By Three+, $c_3 \leq 2$, and observe that row 3 can only be a TWO by having either $(4,3)$ or $(8,3)$ be black. By (2.1), Down $\leq c_8 + 2[c_3 + 3(4) + 3(3)] = 42 + 2c_3 + c_8$. We consider three cases based on the squares marked U and L.

Case 1: The U's are white, and the L's are black. We have $c_8 = 2$, and, by Connectivity, $c_3 = 1$. By (2.1), Across $\leq 2+2[2+3(4)+3(3)] = 48$. Thus, Clues ≤ 94.

Case 2: The U's are black, and the L's are white. The same argument as Case 1 gives Clues ≤ 94.

Case 3: The U's and L's are all white. Here $c_8 = 4$, and none of the rows can be FOUR's. By (2.1), Across $\leq 2+2[2+6(3)] = 42$. Thus, Clues ≤ 92.

The placement in Figure 3d. The squares marked with an X must be black, and $r_4, r_8 \leq 2$. By Connectivity and Three+, we also have $r_3 \leq 2$. By (2.1), Down $\leq c_8 + 2[3(4)+4(3)] = 48 + c_8$.

Case 1: $c_8 \neq 4$. So $c_8 = 2$, and it is impossible to have three top row FOUR's. By (2.1), Across $\leq 2+2[2(2)+3(3)+2(4)] = 44$. Thus, Clues ≤ 94.

Case 2: $c_8 = 4$. Hence, none of the rows can be FOUR's. By (2.1), Across $\leq 2+2[2(2)+5(3)] = 40$. Thus, Clues ≤ 92.

The placement in columns 4, 5, and 6. In this case, we must have $r_1, r_2, r_3, r_5, r_6, r_7 \leq 2$, since there are no squares where any of these rows could be cut into a THREE. By (2.1), Across $\leq 3+2[3+6(2)] = 33$ and Down $\leq 4+2[3(4)+4(3)] = 52$. Thus, Clues ≤ 85.

The placement in columns 1, 5, and 6. Since row 1 is a FOUR, by Three+, columns 2 and 3 cannot have any black squares. Hence $c_2 = c_3 = 1$. Observe that $r_8 \leq 3$. By (2.1), Across $\leq 3+2[3(4)+4(3)] = 51$ and Down $\leq 4+2[3(4)+4(3)] = 44$. Thus, Clues < 96.

The placement in columns 1, 2, and 6. We take advantage of having done the previous cases. If $c_8 = 4$, then Three+ forces $(4, 7)$, $(8, 7)$, and $(12, 7)$ to be black, and we have the case pictured in Figure 2c handled earlier. Hence, we may assume that $c_8 \leq 3$. By Three+, column 3 has no black squares and $c_3 = 1$. By (2.1), Down $\leq 3 + 2[3(4) + 1 + 3(2)] = 47$. Since all of the above cases have been handled, the only possible configuration to consider for three top FOUR's is the 90 degree rotation of this configuration. Thus, Clues ≤ 94.

3.4 2 left FOUR's

Let f be the number of top row FOUR's. By assumption, $f \leq 2$. Since a FOUR in row 8 makes two left FOUR's impossible, we have $r_8 \neq 4$. Similarly, since a FOUR in column 4 makes two left FOUR's impossible, we have $c_4 \neq 4$. Let t be the number of top row TWO's plus the number of left column TWO's.

Claim 1: Clues $\leq r_8 + c_8 + 2f - 2t + 88$. Let t_1 be the number of top row TWO's and t_2 be the number of left column TWO's. So $t = t_1 + t_2$. By (2.1), Across $\leq r_8 + 2[f(4) + (7 - f - t_1)(3) + t_1(2)] = r_8 + 2f - 2t_1 + 42$ and Down $\leq c_8 + 2[2(4) + (5 - t_2)(3) + t_2(2)] = c_8 - 2t_2 + 46$. Of course, Clues = Across + Down.

Claim 2: $c_8 \neq 4$. Suppose to the contrary that we have a FOUR in column 8. Three+ then forces $f \leq 1$. By Claim 1, Clues $\leq r_8 + 2f - 2t + 92$. Since $r_8 < 4$, we must have $f = 1$. The only possible place for the one row FOUR is now row 4. But this makes having two left FOUR's impossible, which is a contradiction.

We now have $r_8, c_8 \leq 3$. Hence, it follows from Claim 1 that 96 clues can be reached only with $f \geq 1$.

Claim 3: Two left FOUR's and at least 96 clues is only possible with:

$$\begin{aligned}
&(A)\ f = 1, t = 0, r_8 = c_8 = 3,\\
&(B)\ f = 2, t = 1, r_8 = c_8 = 3,\\
&(C)\ f = 2, t = 0, r_8 = c_8 = 3,\ \text{or}\\
&(D)\ f = 2, t = 0, r_8 + c_8 = 4.
\end{aligned}$$

This can be seen by considering the ways to get at least 96 on the right-hand side of the inequality in Claim 1. If $f = 1$, then $t = 0$, and $r_8 + c_8 = 6$. So assume that $f = 2$. Since Clues $\leq r_8 + c_8 - 2t + 92$, we must have $t \leq 1$. If $t = 1$, then $r_8 + c_8 = 6$. If $t = 0$, then either $r_8 + c_8 = 6$ or $r_8 + c_8 = 4$.

Claim 4: $c_7 \leq 4$. Suppose to the contrary that we have a FOUR in row 7. Hence, $c_9 = 4$, and $(4, 8)$, $(8, 8)$, and $(12, 8)$ are forced to be black. Since $f \geq 1$, we have row FOUR's whose black squares in column 8 force $c_8 = 2$. By Claim 3, we must have possibility (D) with $t = 0$. However, row 4 can be at most a TWO, which is a contradiction.

The only possible placements of two left FOUR's are now distinguished by the black squares shown in Figure 4.

The placement in Figure 4a. First suppose that U is black. Hence, row 4 can be at most a TWO. By Claim 3, we must have possibility (B) with $t = 1$

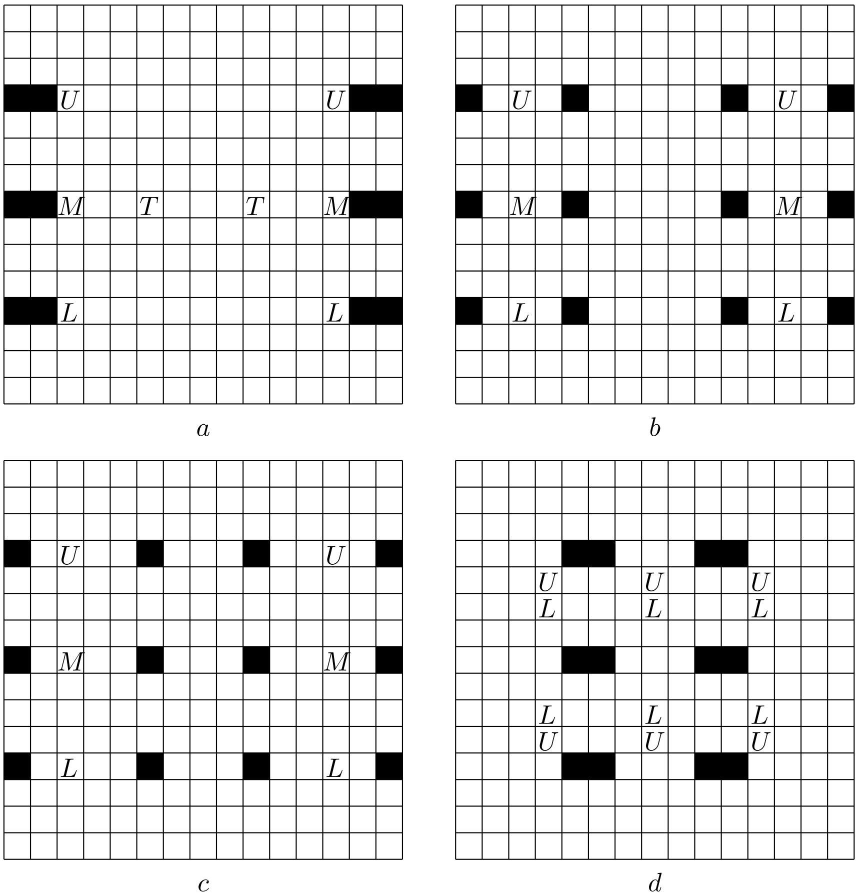


Figure 4: 2 left FOUR's

and $r_8 = 3$. This forces the T's to be black. Since column 3 has to be a THREE, L must be black. However, there is now no way to make column 6 a THREE, which contradicts $t = 1$.

We conclude that U and L are white. So column 3 must be a TWO, with the M's now black. By Claim 3, again we must have possibility (B). However, it is now impossible to make row 8 into a THREE.

The placement in Figure 4b. First suppose that U is black. Hence, row 4 can be at most a TWO. By Claim 3, we must have possibility (B) with $t = 1$ and $r_8 = 3$. This means that column 3 must be a THREE, with L black. However, we now have $r_4 = 1$, which is a contradiction.

We conclude that U and L are white. Therefore, column 3 must be a TWO, with M black. By Claim 3, we must have possibility (B). However, we have $r_8 = 1$, which is impossible.

The placement in Figure 4c. The same argument used for Figure 4b applies here.

The placement in Figure 4d. First suppose that $f = 1$. By Claim 3 possibility (A), we must have $r_8 = c_8 = 3$ and $t = 0$. In order to have a THREE in column 8, the unique top row FOUR must be in either row 5 or row 6. However, to have a THREE in column 7, the top row FOUR must have been in row 5, and the M's must be black. This forces $r_6 \leq 2$, which is a contradiction.

We conclude that $f = 2$ and take advantage of having done the previous cases. Hence, the two row FOUR's must be the 90 degree rotation of the two column FOUR's in this case. This makes the U's and L's black. Now neither row 7 nor column 7 can be cut into a THREE. Hence, $t \geq 2$, which contradicts Claim 3.

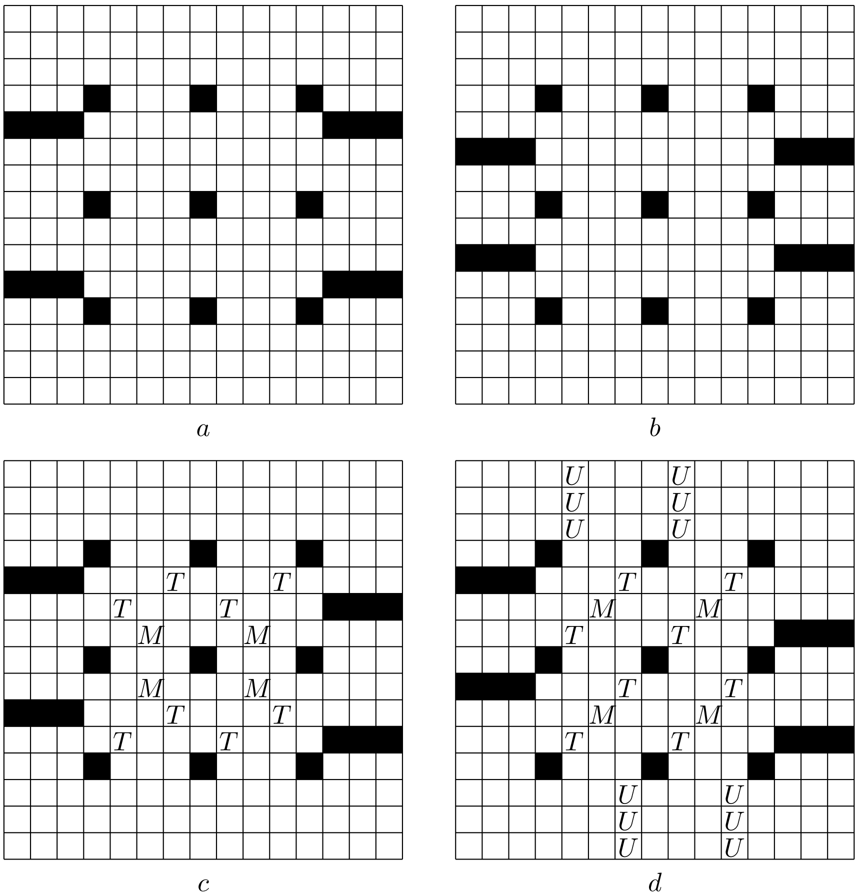


Figure 5: 1 left FOUR

3.5 1 left FOUR

By (2.1), Across $\leq r_8 + 2[4 + 6(3)] = r_8 + 44$ and Down $\leq c_8 + 44$. Hence, Clues $\leq r_8 + c_8 + 88$, and we may assume that $r_8 = c_8 = 4$. These center FOUR's then force the top FOUR to be in row 4 and the left FOUR to be in column 4. The remaining off-center rows and columns must all be THREE's.

By Three+, the rows in which there are black squares cutting column 3 into a THREE also have black squares in columns 1 and 2. The possible cases to consider are distinguished by the black squares shown in Figure 5.

The placement in Figure 5a. It is impossible for row 5 to be a THREE.

The placement in Figure 5b. It is impossible for row 6 to be a THREE.

The placement in Figure 5c. To have THREE's in rows 5 and 6, the T's must be black. To have a THREE in row 7, the M's must be black. However, it is now impossible for column 6 to be a THREE.

The placement in Figure 5d. To have THREE's in rows 5 and 7, the T's must be black. To have a THREE in row 6, the M's must be black. To have a THREE in row 3, the U's must be black. This achieves 96 clues, but Connectivity is violated.

3.6 No left FOUR's

In this final case, we have Across $\leq r_8 + 14(3)$ and Down $\leq c_8 + 14(3)$. Thus, Clues $\leq r_8 + c_8 + 84 \leq 92$.

Q.E.D.