New Negative Latin Square Type Partial Difference Sets in Nonelementary Abelian 2-groups and 3-groups

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Abstract

A partial difference set having parameters $(n^2, r(n-1), n+r^2-3r, r^2-r)$ is called a *Latin square type* partial difference set, while a partial difference set having parameters $(n^2, r(n+1), -n+r^2+3r, r^2+r)$ is called a *negative Latin square type* partial difference set. Nearly all known constructions of negative Latin square partial difference sets are in elementary abelian groups. In this paper, we develop three product theorems that construct negative Latin square type partial difference sets and Latin square type partial difference sets in direct products of abelian groups G and G' when these groups have certain Latin square or negative Latin square type partial difference sets. Using these product theorems, we can construct negative Latin square type partial difference sets in groups of the form $G = (Z_2)^{4s_0} \times (Z_4)^{2s_1} \times (Z_{16})^{4s_2} \times \cdots \times (Z_{2^{2r}})^{4s_r}$ where the s_i are nonnegative integers and $s_0 + s_1 \geq 1$. Another significant corollary to these theorems are constructions of two infinite families of negative Latin square type partial difference sets in 3-groups of the form $G = (Z_3)^2 \times (Z_3)^{2s_1} \times (Z_9)^{2s_2} \times \cdots \times (Z_{3^{2k}})^{2s_k}$ for nonnegative integers s_i. Several constructions of Latin square type PDSs are also given in p-groups for all primes p. We will then briefly indicate how some of these results relate to amorphic association schemes. In particular, we construct amorphic association schemes with 4 classes using negative Latin square type graphs in many nonelementary abelian 2-groups; we also use negative Latin square type graphs whose underlying sets can be elementary abelian 3-groups or nonelementary abelian 3-groups to form 3-class amorphic association schemes.

Keywords: Negative Latin square type partial difference set, Latin square type partial difference set, partial difference set, amorphic association scheme, association scheme, character theory

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1 Introduction

Let G be a finite group of order v with a subset D of order k. Suppose further that the differences $d_1 d_2^{-1}$ for $d_1, d_2 \in D, d_1 \neq d_2$ represent each nonidentity element of D exactly λ times and the nonidentity element of $G - D$ exactly μ times. Then D is called a (v, k, λ, μ)-*partial difference set (PDS)* in G. The survey article of Ma provides a thorough treatment of these sets [6]. A partial difference set having parameters $(n^2, r(n-1), n + r^2 - 3r, r^2 - r)$ is called a *Latin square type PDS*. Similarly, a partial difference set having parameters $(n^2, r(n+1), -n+r^2+3r, r^2+r)$ is called a *negative Latin square type PDS*. Originally, most constructions of both of these types of PDSs were in elementary abelian groups.

Over the past 15 to 20 years, there have been numerous constructions of Latin square type PDSs in nonelementary abelian groups, for instance [1], [4], [5], [6], and [9]. On the other hand, nearly all negative Latin square type PDS constructions had been in elementary abelian groups. Recently, Davis and Xiang constructed an infinite family of negative Latin square type PDSs in some nonelementary abelian 2-groups [2].

In this paper, we will give three product theorems for arbitrary abelian p-groups. The first will show that when one p-group can be partitioned into certain Latin square type PDSs and another p-group into similar negative Latin square type PDSs, then we can construct a partition of negative Latin square type PDSs in the product of the groups. The second product theorem is similar, except in this case both groups are partitioned into Latin square type PDSs and the result is also Latin square type. In the third case, both groups are partitioned into negative Latin square type PDSs and in the product Latin square type PDSs are formed.

Using the first of these product theorems, we will construct 4 negative Latin square type PDSs in many nonelementary abelian groups. Specifically, we can construct such PDSs in groups G of the form $G = (Z_2)^{4s_0} \times (Z_4)^{2s_1} \times (Z_{16})^{4s_2} \times \cdots \times (Z_{2^{2r}})^{4s_r}$ where the s_i are nonnegative integers and $s_0 + s_1 \geq 1$.

Another significant result is in the case of 3-groups. Using the first product theorem we can construct many new negative Latin square PDSs in nonelementary abelian 3-groups. One family generalizes the quadratic form type PDSs that includes the $(81, 20, 1, 6)$-PDS in $(Z_3)^4$ (Theorem 2.6 in [6]). A second parameter set of negative Latin square type PDSs in 3-groups generalizes the known case of the $(81, 30, 9, 12)$-PDS in $(Z_3)^4$.

We will also indicate the usefulness of the second theorem, as it provides constructions of Latin square type PDSs in q-groups for all prime powers $q = p^r$. The third theorem is given for completeness, though at this point it seems less noteworthy due to the fact that negative Latin square type PDSs have been less easily constructed than Latin square type.

Partial difference sets in abelian groups are often studied within the context of the group algebra $\mathbf{Z[G]}$. For a subset D of an abelian group G, $D = \sum_{d \in D} d$ and $D^{(-1)} = \sum_{d \in D} d^{-1}$. The following equations hold for a (v, k, λ, μ)-partial difference set, D, in the abelian group, G, with identity 0:

$$DD^{(-1)} = \lambda P + \mu(G - D - 0) + k0, 0 \notin D.$$

Character theory often is used when studying partial difference sets in abelian groups. A *character* on an abelian group G is a homomorphism from the group to the complex numbers with modulus 1 under multiplication. The *principal character* sends all group elements to 1. The following theorem shows how character sums can be used when studying partial difference sets. See Turyn [10] for a proof of similar results.

Theorem 1.1 *Let G be an abelian group of order v with a subset D of cardinality k such that $k^2 = k + \lambda k + \mu(v - k - 1)$. Then D is a (v, k, λ, μ) partial difference set in G if and only if for every nonprincipal character χ on G, $\chi(D) = \frac{\lambda-\mu\pm\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}$.*

2 Product Theorems for Negative Latin Square Type PDSs and Latin Square Type PDSs

In this section, we will obtain three product theorems that provide the machinery needed for all the results in the paper. Before we proceed with the theorems, it will be convenient to consider a lemma that relates the specific type of PDSs we have with character theory. We will omit the proofs of Theorems 2.2 and 2.3, since they are analogous to the proof of Theorem 2.1.

Lemma 2.1 *Let $q = p^r$ be a prime power and let $e = \pm 1$, and suppose that the abelian group G having order q^{2s} contains subsets ${P_0}^*, P_1, ..., P_{q-1}$ with the following properties:*

(1) ${P_0}^* \cup P_1 \cup \cdots \cup P_{q-1} = G - \{0\}$,

(2) $P_i \cap P_j = \emptyset$ for $i \neq j$,

(3) ${P_0}^*$ is a $(q^{2s}, (q^{s-1} + e)(q^s - e), eq^s + (q^{s-1} + e)^2 - 3e(q^{s-1} + e), (q^{s-1} + e)^2 - e(q^{s-1} + e))$ PDS in G,

(4) For $i \neq 0$, P_i is a $(q^{2s}, q^{s-1}(q^s - e), eq^s + (q^{s-1})^2 - 3e(q^{s-1}), (q^{s-1})^2 - e(q^{s-1}))$ PDS in G.

Let $P_0 = \{0\} \cup {P_0}^*$. Then if χ is a nonprincipal character on G, $\chi(P_i) = eq^s - eq^{s-1}$ for some $0 \leq i \leq p - 1$, and $\chi(P_j) = -eq^{s-1} \forall j \neq i$. When $e = 1$ the PDSs are all of Latin square type and when $e = -1$ the PDSs are all of negative Latin square type.

Proof: Suppose that χ is a nonprincipal character on G. The fact that $\chi(P_j) = \delta(eq^s) - eq^{s-1}$ for $\delta = 0, 1$ is an immediate consequence of Theorem 1.1. Since $\chi(G) = \chi(P_0 \cup P_1 \cup \cdots \cup P_{q-1}) = 0$, we must have exactly one P_i for which $\chi(P_i) = eq^s - eq^{s-1}$.

□

Theorem 2.1 (Product Theorem 1 - Negative Latin × Latin) *Let $q = p^r$ be a prime power, and suppose that abelian groups G and G' have orders q^{2s} and q^{2t} respectively. Suppose further that G contains negative Latin square type PDSs ${P_0}^*, P_1, ..., P_{q-1}$ that satisfy the criteria for Lemma 2.1 and G' posseses Latin square type PDSs $P_0'^*, P_1', ..., P_{q-1}'$ that satisfy the criteria of Lemma 2.1.*

Define $P_0 = {P_0}^* \cup \{0\}$ and $P_0' = P_0'^* \cup \{0\}$.

Then define sets ${D_0}^*, D_1, ..., D_{q-1}$ in $G \times G'$ as follows:

(1) ${D_0}^* = (P_0 \times P_0') \cup (P_1 \times P_1') \cup \cdots \cup (P_{q-1} \times P_{q-1}') - \{(0, 0)\}, D_0 = {D_0}^* \cup \{0, 0\}$

(2) For $i \neq 0, D_i = (P_0 \times P'_i) \cup (P_1 \times P'_{i+1}) \cup \cdots \cup (P_{q-i} \times P'_0) \cup \cdots \cup (P_{q-1} \times P'_{i+q-1})$, where the subscripts are mod q.

The sets D_i have the following properties:

(1) ${D_0}^* \cup D_1 \cup \cdots \cup D_{q-1} = (G \times G') - \{(0,0)\}$,

(2) $D_i \cap D_j = \emptyset$ for $i \neq j$,

(3) ${D_0}^*$ is a $(q^{2(s+t)}, (q^{(s+t)-1} - 1)(q^{s+t} + 1), -q^{s+t} + (q^{s+t-1} - 1)^2 + 3(q^{s+t-1} - 1), (q^{s+t-1} - 1)^2 + (q^{s+t-1} - 1))$ negative Latin square type PDS in $G \times G'$,

(4) For $i \neq 0$, D_i is a $(q^{2(s+t)}, q^{s+t-1}(q^{s+t}+1), -q^{s+t}+(q^{s+t-1})^2+3(q^{s+t-1}), (q^{s+t-1})^2+(q^{s+t-1}))$ negative Latin square type PDS in $G \times G'$.

Proof: It is clear that the sets D_k are disjoint and partition the nonidentity elements of $G \times G'$. Throughout the proof subscripts are mod q.

Let ϕ be a character on $G \times G'$. Then $\phi = \chi \otimes \psi$, where χ is a character on G and ψ is a character on G'.

If ϕ is the principal character, then for $j \neq 0$:

$$\phi({D_0}^*) = |{D_0}^*| = |P_0||P'_0| + (\sum_{i \neq 0} |P_i||P'_i|) - 1 = (q^{2s-1} - q^s + q^{s-1})(q^{2t-1} + q^t - q^{t-1})$$

$$+(q-1)(q^{s-1}(q^s+1))(q^{t-1}(q^t-1) - 1 = (q^{(s+t)-1} - 1)(q^{s+t} + 1).$$

$$\phi(D_j) = |D_j| = |P_0||P'_j| + |P_{-j}||P'_0| + (\sum_{i \neq 0, -j} |P_i||P'_{i+j}|) = (q^{2s-1} - q^s + q^{s-1})(q^{t-1}(q^t - 1)$$

$$+(q^{s-1}(q^s+1)(q^{2t-1} + q^t - q^{t-1}) + (q-2)(q^{s-1}(q^s+1))(q^{t-1}(q^t-1) = q^{s+t-1}(q^{s+t}+1).$$

Now suppose that ϕ is a nonprincipal character on $G \times G'$.

Case 1: χ is principal on G, but ψ is nonprincipal on G'. Then $\chi(P_0) = |P_0| = (q^{s-1} - 1)(q^s + 1) + 1 = q^{2s-1} - q^s + q^{s-1}$ and for $i \neq 0$, $\chi(P_i) = |P_i| = (q^{s-1})(q^s + 1) = q^{2s-1} + q^{s-1}$. ψ will take the values of $-q^{t-1}$ or $q^t - q^{t-1}$ on the sets $P'_0, P'_1, \cdots P'_{p-1}$, and in fact there will be exactly one P_k for which $\psi(P'_k) = q^t - q^{t-1}$ and for all $l \neq k$, $\psi(P_l) = -q^{t-1}$. Then we have:

$$\phi(D_k) = \chi(P_0)\psi(P'_k) + \sum_{m \neq 0} \chi(P_m)\psi(P'_{m+k}) =$$

$$(q^{2s-1} - q^s + q^{s-1})(q^t - q^{t-1}) + (q-1)(q^{2s-1} + q^{s-1})(-q^{t-1}) = -q^{s+t} + q^{s+t-1}$$

and for all $l \neq k$, we get

$$\phi(D_l) = \chi(P_0)\psi(P'_l) + \chi(P_{k-l})\psi(P'_k) + \sum_{m \neq 0} \chi(P_m)\psi(P'_{m+l})$$

$$= (q^{2s-1} - q^s + q^{s-1})(-q^{t-1}) + (q^{2s-1} + q^{s-1})(q^t - q^{t-1}) + (q-2)(q^{2s-1} + q^{s-1})(-q^{t-1}) = q^{s+t-1}.$$

Case 2: χ is nonprincipal on G, but ψ is principal on G'. Then $\psi(P'_0) = |P'_0| = (q^{t-1} + 1)(q^t - 1) + 1 = q^{2t-1} + q^t - q^{t-1}$ and for $i \neq 0$, $\psi(P'_i) = |P'_i|(q^{t-1})(q^t - 1) = q^{2t-1} - q^{t-1}$.

χ will take the values of q^{s-1} or $-q^s + q^{s-1}$ on the sets $P_0, P_1, \cdots P_{p-1}$, and there will be exactly one P_k for which $\chi(P_k) = -q^s + q^{s-1}$ and for all $l \neq k$, $\chi(P_l) = q^{s-1}$. Then we have:

$$\phi(D_{-k}) = \chi(P_k)\psi(P'_0) + \sum_{m\neq k} \chi(P_m)\psi(P'_{m-k}) =$$

$$(q^{2t-1} + q^t - q^{t-1})(-q^s - q^{s-1}) + (q-1)(q^{2t-1} - q^{t-1})(q^{s-1}) = -q^{s+t} + q^{s+t-1}$$

and for all $l \neq k$, we get:

$$\phi(D_{-l}) = \chi(P_l)\psi(P'_0) + \chi(P_k)\psi(P'_{k-l}) + \sum_{m\neq l,k} \chi(P_m)\psi(P'_{m-l}) =$$

$$(q^{2t-1} + q^t - q^{t-1})(q^{s-1}) + (q^{2t-1} - q^{t-1})(-q^s + q^{t-1}) + (q-2)(q^{2t-1} - q^{t-1})(q^{s-1}) = q^{s+t-1}.$$

Case 3: Suppose that both χ and ψ are nonprincipal. Then χ will take the values of q^{s-1} or $-q^s + q^{s-1}$ on the sets $P_0, P_1, \cdots P_{p-1}$, and there will be exactly one P_i for which $\chi(P_i) = -q^s + q^{s-1}$ and for all $j \neq i$, $\chi(P_j) = q^{s-1}$. Also ψ will take the values of $-q^{t-1}$ or $q^t - q^{t-1}$ on the sets $P'_0, P'_1, \cdots P'_{p-1}$, and in fact there will be exactly one P_k for which $\psi(P'_k) = q^t - q^{t-1}$ and for all $l \neq k$, $\psi(P_l) = -q^{t-1}$. Then we have:

$$\phi(D_{k-i}) = \chi(P_i)\psi(P'_k) + \sum_{m\neq i} \chi(P_m)\psi(P'_{m+k-i})$$

$$= (-q^s + q^{s-1})(q^t - q^{t-1}) + (q-1)(q^{s-1})(-q^{t-1}) = -q^{s+t} + q^{s+t-1}$$

and for all $l \neq k-i$, we get

$$\phi(D_l) = \chi(P_i)\psi(P'_{i+l}) + \chi(P_{k-l})\psi(P'_k) + \sum_{m\neq i,k-l} \chi(P_m)\psi(P'_{m+l})$$

$$= (-q^s + q^{s-1})((-q^{t-1}) + (q^{s-1})(q^t - q^{t-1}) + (q-2)(q^{s-1})(-q^{t-1}) = q^{s+t-1}.$$

We have shown that for all nonprincipal characters ϕ, $\phi(D_i) = -\delta_i(q^{s+t}) + q^{s+t-1}$ for $\delta_i = 0, 1$ and $i \neq 0$ and $\phi({D_0}^*) = \phi(D_0) - 1 = -\delta_0(q^{s+t}) + q^{s+t-1} - 1$ for $\delta = 0, 1$. Moreover, we for each nonprincipal character ϕ there will be precisely one D_k such that $\delta_k = 1$. Therefore the result follows from Theorem 1.1.

□

Theorem 2.2 (Product Theorem 2 - Latin × Latin) *Let $q = p^r$ be a prime power, and suppose that abelian groups G and G' have orders q^{2s} and q^{2t} respectively. Suppose further that G contains Latin square type PDSs ${P_0}^*, P_1, ..., P_{q-1}$ that satisfy the criteria for Lemma 2.1 and G' posseses Latin square type PDSs $P'^*_0, P'_1, ..., P'_{q-1}$ that satisfy the criteria of Lemma 2.1.*

Define $P_0 = {P_0}^* \cup \{0\}$ and $P'_0 = P'^*_0 \cup \{0\}$.

Then define sets ${D_0}^*, D_1, ..., D_{q-1}$ in $G \times G'$ as follows:

(1) ${D_0}^* = (P_0 \times P'_0) \cup (P_1 \times P'_1) \cup \cdots \cup (P_{q-1} \times P'_{q-1}) - \{(0,0)\}, D_0 = {D_0}^* \cup \{(0,0)\}$

(2) For $i \neq 0, D_i = (P_0 \times P'_i) \cup (P_1 \times P'_{i+1}) \cup \cdots \cup (P_{q-i} \times P'_0) \cup \cdots \cup (P_{q-1} \times P'_{i+q-1})$, where the subscripts are mod q.

The sets D_i have the following properties:

(1) ${D_0}^* \cup D_1 \cup \cdots \cup D_{q-1} = (G \times G') - \{(0,0)\}$,

(2) $D_i \cap D_j = \emptyset$ for $i \neq j$,

(3) ${D_0}^*$ is a $(q^{2(s+t)}, (q^{(s+t)-1}+1)(q^{s+t}-1), q^{s+t}+(q^{s+t-1}+1)^2-3(q^{s+t-1}+1), (q^{s+t-1}+1)^2-(q^{s+t-1}+1))$ Latin square type PDS in $G \times G'$,

(4) For $i \neq 0$, D_i is a $(q^{2(s+t)}, q^{s+t-1}(q^{s+t}-1), q^{s+t}+(q^{s+t-1})^2-3(q^{s+t-1}), (q^{s+t-1})^2-(q^{s+t-1}))$ Latin square type PDS in $G \times G'$.

Theorem 2.3 (Product Theorem 3 - Negative Latin $\times$ Negative Latin) *Let $q = p^r$ be a prime power, and suppose that abelian groups G and G' have orders q^{2s} and q^{2t} respectively. Suppose further that G contains negative Latin square type PDSs ${P_0}^*, P_1, ..., P_{q-1}$ that satisfy the criteria for Lemma 2.1 and G' posseses negative Latin square type PDSs $P_0'^*, P_1', ..., P_{q-1}'$ that satisfy the criteria of Lemma 2.1.*

Define $P_0 = {P_0}^* \cup \{0\}$ and $P_0' = P_0'^* \cup \{0\}$.

Then define sets ${D_0}^*, D_1, ..., D_{q-1}$ in $G \times G'$ as follows:

(1) ${D_0}^* = (P_0 \times P'_0) \cup (P_1 \times P'_1) \cup \cdots \cup (P_{q-1} \times P'_{q-1}) - \{(0,0)\}, D_0 = {D_0}^* \cup \{(0,0)\}$

(2) For $i \neq 0, D_i = (P_0 \times P'_i) \cup (P_1 \times P'_{i+1}) \cup \cdots \cup (P_{q-i} \times P'_0) \cup \cdots \cup (P_{q-1} \times P'_{i+q-1})$, where the subscripts are mod q.

The sets D_i have the following properties:

(1) ${D_0}^* \cup D_1 \cup \cdots \cup D_{q-1} = (G \times G') - \{(0,0)\}$,

(2) $D_i \cap D_j = \emptyset$ for $i \neq j$,

(3) ${D_0}^*$ is a $(q^{2(s+t)}, (q^{(s+t)-1}+1)(q^{s+t}-1), q^{s+t}+(q^{s+t-1}+1)^2-3(q^{s+t-1}+1), (q^{s+t-1}+1)^2-(q^{s+t-1}+1))$ Latin square type PDS in $G \times G'$,

(4) For $i \neq 0$, D_i is a $(q^{2(s+t)}, q^{s+t-1}(q^{s+t}-1), q^{s+t}+(q^{s+t-1})^2-3(q^{s+t-1}), (q^{s+t-1})^2-(q^{s+t-1}))$ Latin square type PDS in $G \times G'$.

3 Groups with Appropriate Latin Square Type PDSs

There are many groups known to contain partial difference sets that satisfy the hypotheses for Lemma 2.1, especially in the case of Latin square type partial difference sets. For instance, we have the following:

Example 1: In $G = (Z_{p^2})^2$, let $I = pG$ and for $i = 0, ..., p-1$ define the following:

$$A_{0,i} = (< 1, pi > \cup \ < 1, pi+1 > \cup \cdots \cup < 1, pi+(p-1) > \cup < (ip, 1) >) \cap G \setminus I.$$

Then we can define the following:

$$P_0 = A_{0,0} \cup I$$

$$P_i = A_{0,i} \ \forall \ 0 < i \leq p-1.$$

Lemma 3.1 *The sets $P_0, ...P_{p-1}$ are Latin square type PDSs satisfying the criteria to Lemma 2.1.*

Proof Sketch: Let χ be a nonprincipal character on G. If χ has order p^2, then $\chi(A_{0,j}) = p^2-p$ for some j and $\chi(A_{0,i}) = -p$ for all $i \neq j$. Also, $\chi(I) = 0$. So $\chi(P_j) = p^2-p$ and $\chi(P_i) = -p$. If χ has order p, then $\chi(A_{0,i}) = -p$ for all i, while $\chi(I) = p^2$. Thus $\chi(P_0) = p^2 - p$, while $\chi(P_i) = -p$ for $i \neq 0$.

□

Example 2: The sets in $G = (Z_{p^4})^2$ are a bit more complicated; let $I = p^2G$ and for $i = 0, ..., p-1$ define the following:

$$A_{0,i} = (< 1, p^3 i > \cup < 1, p^3 i{+}1 > \cup \cdots \cup < 1, p^3 i{+}(p^3{-}1) > \cup < (ip^3, 1) > \cup < (ip^3{+}p, 1) >$$
$$\cup \cdots \cup < (ip^3 + (p^3 - p), 1) >) \cap G \setminus p^3G.$$

$$A_{1,i} = (< 1, pi > \cup < 1, pi{+}1 > \cup \cdots \cup < 1, pi{+}(p{-}1) > \cup < 1, p^2{+}pi > \cup < 1, p^2{+}pi{+}1 >$$
$$\cup \cdots \cup < 1, p^2{+}pi{+}(p{-}1) > \cup < 1, 2p^2{+}pi > \cup < 1, 2p^2{+}pi{+}1 > \cup \cdots \cup < 1, 2p^2{+}pi{+}(p{-}1) >$$
$$\cup \cdots \cup < 1, (p-1)p^2 + pi > \cup < 1, (p-1)p^2 + pi + 1 > \cup \cdots \cup < 1, (p-1)p^2 + pi + (p-1) >$$
$$\cup < (pi, 1) > \cup < (p^2 + pi, 1) > \cup \cdots \cup < ((p-1)p^2 + pi, 1) >) \cap (p^3G \setminus I).$$

Then we can define the following:

$$P_0 = A_{0,0} \cup A_{1,0} \cup I$$
$$P_i = A_{0,i} \cup A_{1,i} \ \forall\ 0 < i \leq p-1.$$

Lemma 3.2 *The sets $P_0, ...P_{p-1}$ are Latin square type PDSs satisfying the criteria to Lemma 2.1.*

Proof Sketch: Let χ be a nonprincipal character on G. If χ has order p^4, then $\chi(A_{0,j}) = p^4-p^3$ for some j and $\chi(A_{0,i}) = -p^3$ for all $i \neq j$. Also, $\chi(A_{1,k}) = \chi(I) = 0$ for all k. So $\chi(P_j) = p^4 - p^3$ and $\chi(P_i) = -p^3$. If χ has order p^3, then $\chi(A_{1,j}) = p^4 - p^3$ for some j and $\chi(A_{1,i}) = -p^3$ for all $i \neq j$. Also, $\chi(A_{0,k}) = \chi(I) = 0$ for all k. So $\chi(P_j) = p^4 - p^3$ and $\chi(P_i) = -p^3$. If χ has order p^2, then $\chi(A_{0,i}) = 0$ and $\chi(A_{1,i}) = -p^3$ for all i, while $\chi(I) = p^4$. Thus $\chi(P_0) = p^4 - p^3$, while $\chi(P_i) = -p^3$ for $i \neq 0$. If χ has order p, then $\chi(A_{0,i}) = -p^5$ and $\chi(A_{1,i}) = p^5 - p^3$ for all i, while $\chi(I) = p^4$. Thus $\chi(P_0) = p^4 - p^3$, while $\chi(P_i) = -p^3$ for $i \neq 0$.

□

These sets $A_{i,j}$ are defined more generally for $(Z_{p^{2r}})^{2t}$ in [8], and can be used to form the desired PDSs for our product theorems. The remainder of this section is taken from [8] to give us the following result, which will be subsequently used in conjunction with Theorem 2.1 to give us the main results from sections 4 and 5.

Theorem 3.1 *For $q = p^t$, every group of the form $(Z_{p^{2r}})^{2t}$ has Latin square type partial difference sets $P_0, P_1, ..., P_{p^t-1}$ that satisfy the hypotheses of Lemma 2.1.*

To prove Theorem 3.1 we will use the structure of Galois rings. If $\phi_1(x)$ is a primitive irreducible polynomial of degree t over F_p, then $F_p[x]/\langle\phi_1(x)\rangle$ is a finite field of order p^t. Hensel's lemma states that there is a unique primitive irreducible polynomial $\phi_s(x)$ over Z_{p^s} so that $\phi_s(x) \equiv \phi_1(x) \bmod p$ and with a root ω of $\phi_s(x)$ satisfying $\omega^{p^t-1} = 1$. Then $Z_{p^s}[\omega]$ is the *Galois extension of Z_{p^s} of degree t*, and furthermore $Z_{p^s}[\omega]$ is called a Galois ring denoted $GR(p^s, t)$. Clearly the additive group of $GR(p^s, t)$ is isomorphic to $(Z_{p^s})^t$. See [7] for a detailed description of Galois rings.

An important subset of $GR(p^s, t)$ is the Teichmuller set $\mathcal{T} = \{0, 1, \omega, \omega^2, ..., \omega^{p^t-2}\}$, which can be viewed as the set of all solutions to the polynomial $x^{p^t} - x$ over $GR(p^s, t)$. A canonical way of uniquely expressing an element of $GR(p^s, t)$ is:

$$\alpha = \alpha_0 + p\alpha_1 + p^2\alpha_2 + ... + p^{s-1}\alpha_{s-1}$$

where $\alpha_i \in \mathcal{T}$. We see that the invertible elements are those with $\alpha_0 \neq 0$, and if we take the natural projection (modulo p reduction) from $GR(p^s, t)$ to $GF(p^t)$, then $\mathcal{T}$ maps onto $GF(p^t)$; this projection is given by $\pi(\alpha) = \alpha_0 \bmod p$ in the representation above.

We define subgroups in $GR(p^s, t) \times GR(p^s, t)$ by the following:

$$S_{i_s,i_{s-1},...,i_2,i_1} = \{\alpha, (i_1 + pi_2 + p^2 i_3 + ...p^{s-2} i_{s-1} + p^{s-1} i_s)\alpha) | \alpha \in GR(p^s, t)\}$$

$$S_{i_s,i_{s-1},...,i_2,\infty} = \{((pi_2 + p^2 i_3 + ...p^{s-2} i_{s-1} + p^{s-1} i_s)\alpha, \alpha) | \alpha \in GR(p^s, t)\}$$

In the above, the subscripts $i_j \in \mathcal{T}$.

Now we will require that the exponent be even, so let $R = GR(p^{2r}, t)$. Let M be the additive group of $R \times R$, and let $I = p^r M$. We are ready to define the sets $A_{i,j}$ that will be used to form our partial difference sets.

$$A_{0,j} = \bigcup_{y_{2r-1},y_{2r-2},...,y_1} S_{a_j,y_{2r-1},y_{2r-2},...,y_1} \cap (M \setminus pM)$$

$$A_{i,j} = \bigcup_{y_{2r-i},...,y_{2r-2i+1},y_{2r-2i-1},...,y_1} S_{0,0,...,0,y_{2r-i},...,y_{2r-2i+1},a_j,y_{2r-2i-1},...,y_1} \cap (p^i M \setminus p^{i+1} M), \ \ i \in \{1, 2, ..., r-1\},$$

where the a_j are fixed elements in the Teichmuller set $\mathcal{T}$ and the y_l are allowed to vary over all possible values, so $y_l \in \mathcal{T}$ for $l \neq 1$ and $y_1 \in \mathcal{T} \cup \infty$. The proof of the following lemma on the sets $A_{i,j}$ can be found in [8], and is the key component in constructing our Latin square type PDSs.

Lemma 3.3 *Let χ be a character on $M = (Z_{p^{2r}})^{2t}$. Then χ is order p^k for some integer k with $0 \leq k \leq 2r$. If $k \neq 2r - l$, then $\chi(A_{l,i}) = \chi(A_{l,j})$ $\forall i, j$. If $k = 2r - l$ then $\chi(A_{l,j'}) = p^{2rt} - p^{(2r-1)t}$ for some j' and $\chi(A_{l,j}) = -p^{(2r-1)t}$ for all $j \neq j'$.*

The following sets $P_0^*, P_1, ..., P_{p^t-1}$ as given below are the Latin square type PDSs in $(Z_{p^{2r}})^{2t}$ that satisfy the criteria to Lemma 2.1.

$$P_0 = (\bigcup_{i=0}^{r-1} A_{0,i}) \cup I,$$

$$P_j = (\bigcup_{i=0}^{r-1} A_{j,i}), j = 1, 2, ..., p^t - 1.$$

Proof (Theorem 3.1): The sets P_j partition the elements of $R \times R$.

If χ is a character of order p^k for $0 < k \leq r$, then by Lemma 3.3 for every i it will be the case that $\chi(A_{i,j}) = \chi(A_{i,j'})$ for all j, j'. χ will be trivial on I, so $\chi(I) = p^{2rt}$. Since $\chi(M) = \chi(\bigcup_{j=0}^{p^t-1} P_j) = 0$, a little algebra gives that $\chi(P_0) = p^{2rt} - p^{(2r-1)t}$ and for $j \neq 0$, $\chi(P_j) = -p^{(2r-1)t}$.

If χ is a character of order p^k for $r < k \leq 2r$, then by Lemma 3.3 for every l with $l+k \neq 2r$ it will be the case that $\chi(A_{l,j}) = \chi(A_{l,j'})$ for all j, j'. Further, $\chi(I) = 0$. For $l = 2r - k$ we will have by Lemma 3.3 that $\chi(A_{l,j'}) = p^{2rt} - p^{(2r-1)t}$ for some j' and $\chi(A_{l,j}) = -p^{(2r-1)t}$ for all $j \neq j'$. It follows that $\chi(P_{j'}) = p^{2rt} - p^{(2r-1)t}$ and $\chi(A_{l,j}) = -p^{(2r-1)t}$ for all $j \neq j'$. The result follows.

□

4 The Prime 2 Case

In the case where we use $p = 2$ in Theorem 2.1 or Theorem 2.2, we get a weaker version of the well-known result on reversible Menon-Hadamard difference sets since it is restricted to 2-groups. See [6] for instance. However, when we use $q = 4$ we are able to generalize the negative Latin square type PDSs of Davis and Xiang [2,3].

Let $G = (Z_2)^4$ or $(Z_4)^2$, as both have 3 disjoint $(16, 5, 0, 2)$-PDSs P_1, P_2, and P_3. In $G = (Z_2)^4$ we have:

$$P_1 = \{(1,0,0,0), (0,0,1,0), (0,1,0,1), (1,0,0,1), (0,1,1,0)\},$$

$$P_2 = \{(0,1,0,0), (0,0,0,1), (1,1,1,1), (0,1,1,1), (1,1,0,1)\},$$

$$P_3 = \{(1,1,0,0), (0,0,1,1), (1,0,1,0), (1,1,1,0), (1,0,1,1)\},$$

and we have $P_0 = \{(0,0,0,0)\}$ and $P_0^* = \emptyset$. In $(Z_4)^2$ we have:

$$P_1 = \{(2,0), (0,1), (0,3), (1,1), (3,3)\},$$

$$P_2 = \{(1,0), (3,0), (0,2), (1,3), (3,1)\},$$

$$P_3 = \{(1,2), (3,2), (2,1), (2,3), (2,2)\},$$

and then $P_0 = \{(0,0)\}$ and $P_0^* = \emptyset$. In both cases we have negative Latin square type PDSs that satisfy the criteria of Lemma 2.1 for $q = 4$ and $s = 1$. In this particular case, P_0^* is a degenerate PDS and of course $\chi(P_0) = 1$ for all characters χ on G. We can also find four Latin square type PDSs that satisfy the criteria for Lemma 2.1 in both $G' = (Z_2)^4$ or $(Z_4)^2$. In $(Z_2)^4$ we take:

$$P'_1 = \{(1,0,0,0), (0,1,0,0), (1,1,0,0)\},$$

$$P'_2 = \{(0,0,1,0), (0,0,0,1), (0,0,1,1)\},$$

$$P'_3 = \{(1,0,1,0), (0,1,0,1), (1,1,1,1)\},$$

$$P_0'^* = G' \setminus (P_1' \cup P_2' \cup P_3' \cup \{(0,0,0,0)\}).$$

In $G' = (Z_4)^2$ we take:

$$P_1' = \{(1,0),(2,0),(3,0)\},$$
$$P_2' = \{(0,1),(0,2),(0,3)\},$$
$$P_3' = \{(1,1),(2,2),(3,3)\},$$
$$P_0'^* = G' \setminus (P_1' \cup P_2' \cup P_3' \cup \{(0,0)\}).$$

These sets allow us to recursively use Theorem 2.1 to give us the following result.

Theorem 4.1 *For nonnegative integers s, t with $s + t \geq 1$, every group of the form $(Z_2)^{4t} \times (Z_4)^{2s}$ has four negative Latin square type PDSs that satisfy the criteria to Lemma 2.1*

Setting $p = 2$ and $t = 2$ in Theorem 3.1 gives us that there are Latin square type PDSs P_0', P_1', P_2', and P_3' that satisfy the criteria to Lemma 2.1 in all groups of the form $(Z_{2^{2r}})^4$. Therefore we can combine Theorem 2.1 with Theorem 3.1 to give us the following result.

Lemma 4.1 *Let $K = (Z_2)^4$ or $(Z_4)^2$. Then every group of the form $K \times (Z_{2^{2r}})^4$ has negative Latin square type PDSs that satisfy the criteria of Lemma 2.1.*

We can then use the preceding lemma and recursive use of Theorem 2.1 to give us the following siginificant result.

Theorem 4.2 *For nonnegative integers s_0, s_1 with $s_0 + s_1 \geq 1$, every group of the form $(Z_2)^{4s_0} \times (Z_4)^{2s_1} \times (Z_{16})^{4s_2} \times \cdots \times (Z_{2^{2r}})^{4s_r}$, where the s_i are nonnegative integers, has 4 negative Latin square type PDSs D_0^*, D_1, D_2, D_3 that satisfy the criteria to Lemma 2.1. If $|G| = 2^{2M}$, then D_0^* is a $(2^{2M}, (2^{M-2}-1)(2^M+1), -2^M + (2^{M-2}-1)^2 + 3(2^{M-2}-1), (2^{M-2}-1)^2 + (2^{M-2}-1))$-PDS while for $i = 1,2,3$, D_i is a $(2^{2M}, 2^{M-2}(2^M+1), -2^M + (2^{M-2})^2 + 3(2^{M-2}), (2^{M-2})^2 + (2^{M-2}))$-PDS.*

5 New Negative Latin Square Type PDSs in 3-groups

In the specific case where $q = 3$ in Theorem 2.1, we have a similar thing as what occurs for $q = 4$. If in the subgroup $(Z_3)^2$ we define sets $P_0{}^* = \emptyset$, $P_1 =< 1,0 > \cup < 0,1 > -\{0,0\}$, and $P_2 =< 1,1 > \cup < 1,2 > -\{0,0\}$ we can utilize the same construction as in Theorem 2.1 to generate negative Latin square type PDSs in any group of the form $(Z_3)^2 \times G$, where G is a 3-group that contains Latin square type PDSs satisfying Lemma 2.1.

To demonstrate the construction more clearly, we provide the following examples below.

Example 1: Let $G = G' = (Z_3)^2$, so we have $q = 3$ and $s = t = 1$. In G we define sets $P_0{}^* = \emptyset$, $P_1 =< (1,0) > \cup < (0,1) > -\{(0,0)\}$, and $P_2 =< (1,1) > \cup < (1,2) > -\{(0,0)\}$. In G', we have the Latin square type PDSs $P_0'^* =< (0,1) > \cup < (1,0) > -\{(0,0)\}$, $P_1' =< (1,1) > -\{(0,0)\}$, and $P_2' =< (1,2) > -\{(0,0)\}$. Then we define $P_0 = P_0{}^* \cup \{(0,0)\}$ and $P_0' = P_0'^* \cup \{(0,0)\}$. Then define sets $D_0{}^*, D_1, D_2$ in $G \times G'$ as follows:

$$D_0{}^* = (P_0 \times P_0') \cup (P_1 \times P_1') \times (P_2 \times P_2') - \{0,0\},$$

$$D_1 = (P_0 \times P_1') \cup (P_1 \times P_2') \times (P_2 \times P_0'),$$
$$D_2 = (P_0 \times P_2') \cup (P_1 \times P_0') \times (P_2 \times P_1').$$

${D_0}^*$ is an $(81, 20, 1, 6)$-PDS, while D_1 and D_2 are both $(81, 30, 9, 12)$-PDSs. Moreover, they satisfy the hypotheses of Lemma 2.1 and can be used in Theorem 2.1. Using this approach recursively gives us three negative Latin square type PDSs in $(Z_3)^{2t}$ for all integers $t \geq 2$. ${D_0}^*$ is a $(3^{2t}, (3^{t-1}-1)(3^t+1), -3^t+(3^{t-1}-1)^2+3(3^{t-1}-1), (3^{t-1}-1)^2+(3^{t-1}-1))$ negative Latin square type PDS in $G \times G'$ while both D_1 and D_2 are $(3^{2t}, (3^{t-1})(3^t+1), -3^t+(3^{t-1})^2+3(3^{t-1}), (3^{2t-1})^2+(3^{2t-1}))$ negative Latin square type PDSs.

Example 2: Let $G = (Z_3)^2$ and $G' = (Z_9)^2$, so we have $q = 3$, $s = 1$, and $t = 2$. In G we define sets ${P_0}^* = \emptyset$, $P_1 = <(1,0)> \cup <(0,1)> -\{(0,0)\}$, and $P_2 = <(1,1)> \cup <(1,2)> -\{(0,0)\}$. In G', we have the Latin square PDSs ${P_0'}^* = <(0,1)> \cup <(1,0)> \cup <(1,1)> \cup <(1,2)> -\{(0,0)\}$, $P_1' = <(3,1)> \cup <(1,3)> \cup <(1,4)> \cup <(1,5)> -3G'$ and $P_2' = <(6,1)> \cup <(1,6)> \cup <(1,7)> \cup <(1,8)> -3G'$. The sets ${P_0'}^*, P_1', P_2'$ are PDSs that satisfy the criteria of Lemma 2.1. Then define sets ${D_0}^*, D_1, D_2$ in $G \times G'$ as follows:

$${D_0}^* = (P_0 \times P_0') \cup (P_1 \times P_1') \times (P_2 \times P_2') - \{0, 0\},$$
$$D_1 = (P_0 \times P_1') \cup (P_1 \times P_2') \times (P_2 \times P_0'),$$
$$D_2 = (P_0 \times P_2') \cup (P_1 \times P_0') \times (P_2 \times P_1').$$

${D_0}^*$ is an $(729, 224, 61, 72)$-PDS, while D_1 and D_2 are both $(729, 252, 81, 90)$-PDSs. Moreover, they satisfy the hypotheses of Lemma 2.1 and can be used in Theorem 2.1. Using this approach recursively gives us three negative Latin square type PDSs in $(Z_3)^{2s} \times (Z_9)^{2t}$ for all integers $s, t \geq 1$ that satisfy the criteria to Lemma 2.1.

Putting this all together, we get the following more general corollary to Theorem 2.1.

Corollary 5.1 *There exist negative Latin square type PDSs of two different parameters in all groups of the form* $G = (Z_3)^2 \times (Z_3)^{2s_1} \times (Z_9)^{2s_2} \times \cdots \times (Z_{3^{2k}})^{2s_k}$ *where the* s_i *are nonnegative integers. If* $|G| = 3^{2m}$*, then the parameters are* $(3^{2m}, (3^{m-1}-1)(3^m+1), -3^m+(3^{m-1}-1)^2+3(3^{m-1}-1), (3^{m-1}-1)^2+(3^{m-1}-1)$ *and* $(3^{2m}, (3^{m-1})(3^m+1), -3^m+(3^{m-1})^2+3(3^{m-1}), (3^{m-1})^2+(3^{m-1}))$.

Notice that the first set of parameters is known, for instance Theorem 2.6 in [6]. The second set of parameters is a generalization of the $(81, 30, 9, 12)$-PDS in $(Z_3)^4$. For both sets of parameters, we have many new examples as all the previous constructions were in elementary abelian groups.

6 New Latin Square Type PDSs in p-groups

Putting together Theorem 2.2 with Theorem 3.1 yields the following corollary that works for all primes p.

Corollary 6.1 *For all primes p there exist Latin square type PDSs* ${D_0}^*, D_1, ..., D_{p-1}$ *that satisfy the criteria of Lemma 2.1 in all groups of the form* $G = (Z_p)^{2s_1} \times (Z_{p^2})^{2s_2} \times \cdots \times (Z_{p^{2k}})^{2s_k}$ *where the* s_i *are nonnegative integers.*

We can use these PDSs to generate more PDSs using the group G as well by using the methods of [1] (their Theorem 3.2).

Corollary 6.2 *Let $G = (Z_p)^{2s_1} \times (Z_{p^2})^{2s_2} \times \cdots \times (Z_{p^{2k}})^{2s_k}$ where the s_i are nonnegative integers, and let $|G| = n^2$. Let $i_1, i_2, ...i_l \subset \{1, ..., p-1\}$, $1 \leq l \leq p-1$. Then $D_{i_1} \cup D_{i_2} \cup \cdots D_{i_l}$ is an $(n^2, r_1(n-1), n + {r_1}^2 - 3r_1, {r_1}^2 - r_1)$-PDS in G. We have $n = p^{s_1+2s_2+\cdots 2ks_k} = p^M$ and $r_1 = l(p^{M-1})$. Also $D_0^* \cup D_{i_1} \cup D_{i_2} \cup \cdots D_{i_l}$ is an $(n^2, r_2(n-1), n + {r_1}^2 - 3r_1, {r_1}^2 - r_1)$-PDS in G where $r_2 = (l+1)(p^{M-1}) + 1$.*

Proof: We prove the first set of parameters. For any nonprincipal character χ on G, we know that $\chi(D_i) = p^M - p^{M-1}$ for some $0 \leq i \leq p-1$, and $\chi(D_j) = -p^{M-1} \forall j \neq i$. If $i \in i_1, i_2, ...i_l$, then $\chi(D_{i_1} \cup D_{i_2} \cup \cdots D_{i_l}) = p^M - lp^{M-1}$. Otherwise, $\chi(D_{i_1} \cup D_{i_2} \cup \cdots D_{i_l}) = lp^{M-1}$. The result follows from Theorem 1.1. The proof for the second set of parameters is similar.

□

7 Amorphic Association Schemes

While the results in this paper have been in the area of partial difference sets, they fit quite nicely into the context of association schemes. Let X be a finite set. An *association scheme* with d classes on X consists of sets (binary relations) $R_0, R_1, \cdots R_d$ that partition $X \times X$ and further that:

(1) $R_0 = \{(x, x) | x \in X\}$ (the identity relation);

(2) R_l is symmetric $\forall l$;

(3) $\forall i, j, k \in \{0, 1, 2, \cdots d\}$ there is an integer p_{ij}^k such that given any pair $(x, y) \in R_k$,

$$|\{z \in X | (x, z) \in R_i \text{ and } (z, y) \in R_j\}| = p_{ij}^k.$$

Each of the symmetric relations R_l can be interpreted as an undirected graph with vertex set X and edge set R_l, $G_l = (X, R_l)$ for all l. Then we can think of an association scheme as a decomposition of the complete graph with X as the vertex set into the graphs G_l with the property that for $i, j, k \in \{1, 2, \cdots d\}$ and for every $xy \in E(G_k)$,

$$|\{z \in X | xz \in E(G_i) \text{ and } zy \in E(G_j)\}| = p_{ij}^k,$$

where $E(G_i)$ denotes the edge set of graph G_i. These graphs G_i are called the *graphs* of the association scheme. A *strongly regular graph* is an association scheme with two classes.

Given an association scheme, we can take unions of classes to produce graphs with larger edge sets, a so called *fusion*. Fusions are not necessarily themselves association schemes, but when a particular association scheme does have the property that any of its fusions is also an association scheme we call the scheme *amorphic*. For a thorough treatment of association schemes, see [12].

Partial difference sets give rise to strongly regular Cayley graphs. When we partition the groups as in Lemma 2.1 into partial difference sets, we can think of this as partitioning the complete graph with vertex set the group elements into strongly regular graphs of Latin square type or negative Latin square type. A theorem of Van Dam [11] will allow us to put the results in this paper in the context of association schemes.

Theorem 7.1 *Let $\{G_1, G_2, \cdots G_d\}$ be an edge-decomposition of the complete graph on a set X, where each G_i is strongly regular. If the G_i are all of Latin square type or all of negative Latin square type, then the decomposition is a d-class amorphic association scheme on X.*

Every result in this paper involves a partitioning of the p-group G into Latin square type or negative Latin square type partial difference sets. Therefore, the results would fit well into the context of association schemes. In particular, using Theorem 4.2 and Corollary 5.1 in conjunction with Theorem 7.1 we have the following corollaries.

Corollary 7.1 *The strongly regular Cayley graphs associated to the negative Latin square PDSs in Theorem 4.2 form a 4-class amorphic association scheme on $G = (Z_2)^{4s_0} \times (Z_4)^{2s_1} \times (Z_{16})^{4s_2} \times \cdots (Z_{2^{2r}})^{4s_r}$, where the s_i are nonnegative integers and $s_0 + s_1 \geq 1$.*

Corollary 7.2 *The strongly regular Cayley graphs associated to the negative Latin square PDSs in Corollary 5.1 form a 3-class amorphic association scheme on $G = (Z_3)^2 \times (Z_3)^{2s_1} \times (Z_9)^{2s_2} \times \cdots \times (Z_{3^{2k}})^{2s_k}$.*

In [3], Davis and Xiang constructed 4-class amorphic association schemes in nonelementary abelian 2-groups using negative Latin square type partial difference sets. Corollary 7.1 introduces 4-class amorphic association schemes associated with some groups that are not a part of their construction, in particular all of those groups with characteristic greater than 4. It is believed that Corollary 7.2 gives the first amorphic association scheme derived from negative Latin square type PDSs in nonelementary abelian p-groups for $p \neq 2$. Notice that it includes both elementary abelian and nonelementary abelian examples.

The product theorems in this paper were already shown to be useful in the construction of both Latin square type partial difference sets and negative Latin square partial difference sets. Moreover, they also yielded amorphic assocation schemes derived with negative Latin square type graphs in nonelementary abelian 3-groups. It seems quite likely that they will lead to further results as there are numerous constructions of both Latin square type and negative Latin square type PDSs that could be analyzed for use with Theorems 2.1, 2.2, and 2.3.

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