Summing Up the Euler ϕ Function

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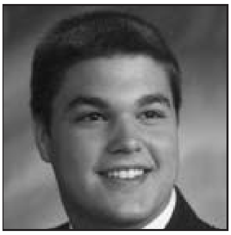

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Asking open-ended questions in class has many potential benefits: it can give students a chance to come to a result on their own, it can lead to new questions and deeper investigations of important concepts, and it can point to gaps in students' understanding. Sometimes it even stimulates new research. Number theory, a subject in which conjectures can be easy to make but often difficult to settle, is particularly well-suited to asking students what they see in a particular example. This paper is the result of an innocent question and an unexpected answer from a number theory course. When the authors first submitted this article, they believed the work to be original. As it turns out, the majority of the work was a rediscovery of previous results, although the research experience was still a very positive one.

One of the standard topics in a first course in number theory is the Euler ϕ function, with $\phi(n)$ defined as the number of positive integers less than n and relatively prime to n. A famous theorem involving ϕ is Gauss's theorem that the sum of $\phi(d)$ over the divisors d of n is n; that is,

$$\sum_{d|n} \phi(d) = n.$$

For example, the divisors of 15 are 1, 3, 5, and 15, and $\phi(1) + \phi(3) + \phi(5) + \phi(15) = 1 + 2 + 4 + 8 = 15$. Gauss proved this theorem by introducing the sets

$S_d = \{m | 1 \leq m \leq n \text{ and } \gcd(m, n) = d\}$. Thus, for $n = 15$,

$$\begin{aligned} S_1 &= \{1, 2, 4, 7, 8, 11, 13, 14\} && \text{(8 elements)} \\ S_3 &= \{3, 6, 9, 12\} && \text{(4 elements)} \\ S_5 &= \{5, 10\} && \text{(2 elements)} \\ S_{15} &= \{15\} && \text{(1 element).} \end{aligned}$$

We note that each integer from 1 to 15 appears in exactly one S_d. Gauss went on to show that the sizes of the S_d are the same as the values of $\phi(d)$, albeit in reverse order, and from this the result follows.

The first author gave this example to a class and asked if anyone saw a pattern. He was hoping that someone would stumble onto Gauss's theorem. Instead, the second author noted that the size of each set could be found by taking ϕ of the size of the previous set. That is, $\phi(8) = 4$, $\phi(4) = 2$, and $\phi(2) = 1$. As a result, $15 = \phi(15) + \phi(\phi(15)) + \phi(\phi(\phi(15))) + \phi(\phi(\phi(\phi(15))))$. This was a surprise, and led to the obvious question: For which numbers does this happen?

Background

All variables in this paper will represent positive integers. Two facts about the Euler ϕ function make evaluating $\phi(n)$ straightforward. First, if p is prime, then

$$\phi(p^k) = p^k - p^{k-1}.$$

Second, ϕ is multiplicative; that is, if m and n are relatively prime, then $\phi(mn) = \phi(m)\phi(n)$. For example, $\phi(60) = \phi(2^2)\phi(3)\phi(5) = (4-2)(3-1)(5-1) = 16$. As usual, we let $\phi^1(n) = \phi(n)$ and $\phi^i(n) = \phi(\phi^{i-1}(n))$. Hence we can iterate ϕ to create the sequence $\{n, \phi(n), \phi^2(n)), \ldots\}$. Parts of some such sequences are shown in Figure 1, where $a \rightarrow b$ if $\phi(a) = b$.

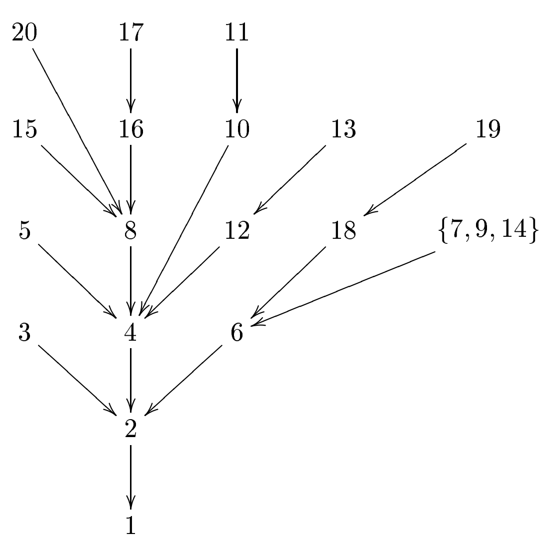

Figure 1. Iteration of ϕ.

Since $\phi(n)$ is less than n, such a sequence is strictly decreasing and reaches 1 after a finite number of steps.

Following Pillai [**7**], let $R(n)$ denote the smallest integer k such that $\phi^k(n) = 1$. That is, $R(n)$ is the number of steps it takes the sequence beginning with n to reach 1. Toward answering our question, we make two more definitions; they are the focus of this paper.

Define $\Phi(n)$ by

$$\Phi(n) = \sum_{i=1}^{R(n)} \phi^i(n).$$

Looking back at Figure 1, $\Phi(n)$ can be viewed as the sum of the numbers on the path in the tree from n to 1 (including 1). Thus $\Phi(20) = \Phi(16) = \Phi(15) = 8 + 4 + 2 + 1 = 15$. Note also that since $R(1) = 0$, $\Phi(1) = 0$.

We call n a *perfect totient number (PTN)* if $\Phi(n) = n$.

Thus our central question becomes: which numbers are PTNs? We found one, 15, in the introduction. The following Mathematica command will generate all those less than 100000 and they are listed in the table that follows.

```
For[n=2, n<100000, n++, k=0; t=n;
    While[(t=EulerPhi[t])!>1, k=k+t];
    If [k+1-n==0, Print[n]]]
```

Table 1. Perfect totient numbers less than 100000.

3	$183 = 3 \cdot 61$	$2187 = 3^7$	$8751 = 3 \cdot 2917$
$9 = 3^2$	$243 = 3^5$	$2199 = 3 \cdot 733$	$15723 = 3^2 \cdot 1747$
$15 = 3 \cdot 5$	$255 = 3 \cdot 5 \cdot 17$	$3063 = 3 \cdot 1021$	$19683 = 3^9$
$27 = 3^3$	$327 = 3 \cdot 109$	$4359 = 3 \cdot 1453$	$36759 = 3 \cdot 12253$
$39 = 3 \cdot 13$	$363 = 3 \cdot 11^2$	$4375 = 5^4 \cdot 7$	$46791 = 3^3 \cdot 1733$
$81 = 3^4$	$471 = 3 \cdot 157$	$5571 = 3^2 \cdot 619$	$59049 = 3^{10}$
$111 = 3 \cdot 37$	$729 = 3^6$	$6561 = 3^8$	$65535 = 3 \cdot 5 \cdot 17 \cdot 257$

This table reveals a few families of PTNs, as well as some oddballs. Before we look more deeply at PTNs, we give four lemmas that will be helpful. Proofs of the first two are straightforward uses of the multiplicative property of $\phi(n)$ and the third follows immediately from the definition of Φ. It follows from the first three lemmas that

$$\Phi(2^i 3^j) = \sum_{m=0}^{j-1} 2^i 3^m + \sum_{n=0}^{i-1} 2^n,$$

and a little algebra gives the fourth.

Lemma 1. *If k is odd and $i > 0$, then $\phi(2^i k) = 2^{i-1}\phi(k)$. In particular, if n is even, $\phi(n) \le \frac{1}{2}n$.*

Lemma 2. *If $i, j > 0$, then $\phi(2^i 3^j) = 2^i 3^{j-1}$.*

Lemma 3. *If $n > 1$, $\Phi(n) = \phi(n) + \Phi(\phi(n))$.*

Lemma 4. *If $i, j > 0$, then $\Phi(2^i 3^j) = 2^{i-1}(3^j + 1) - 1$.*

Perfect totient numbers

In this section we present several theorems that explain many of the PTNs listed above and provide a few more examples. We conclude the section with some open questions.

Theorem 1. *If n is even, then $\Phi(n) < n$.*

Proof. (By induction.) Note that $\Phi(2) = 1$. Now let k be even and assume that the theorem is true for all even $j < k$. Then $\Phi(k) = \phi(k) + \Phi(\phi(k)) < \phi(k) + \phi(k)$ by the induction hypothesis, since $\phi(k)$ is even. By Lemma 1, $\phi(k) \leq \frac{1}{2}k$, and the result follows. ■

Corollary 1. *For all n, $\Phi(n) < 2\phi(n)$.*

Proof. Note that $\Phi(2) = 1$, and if $n > 2$, then $\phi(n)$ is even. Thus $\Phi(\phi(n)) < \phi(n)$, and by Lemma 3 we are done. ■

Thus all PTNs must be odd. The next theorem gives us our first family of these numbers.

Theorem 2. *Let n be a prime power, $n = p^k$. Then*

$$\begin{aligned} \Phi(n) &> n && \text{if } p > 3; \\ \Phi(n) &= n && \text{if } p = 3; \\ \Phi(n) &= n - 1 && \text{if } p = 2. \end{aligned}$$

Proof. Here we will use the fact that, as long as $n > 2$, $\Phi(n) = \phi(n) + \phi(\phi(n)) +$ possibly more terms. Then if $p > 3$, $\Phi(p) \geq p - 1 + \phi(p-1) > p$. If $k \geq 2$, recall that $\phi(p^k) = p^k - p^{k-1}$, so

$$\phi(\phi(p^k)) = \phi(p-1)(p^{k-1} - p^{k-2}) \geq 2(p^{k-1} - p^{k-2}).$$

So for $p > 3$, some careful algebra gives us $\phi(p^k) \geq p^k + p^{k-1} - 2p^{k-2} > p^k$. Now if $p = 3$, $\phi(p^k) = 2 \cdot 3^{k-1}$. By Lemma 2, $\Phi(2 \cdot 3^{k-1}) = 3^{k-1}$, so

$$\Phi(3^k) = 2 \cdot 3^{k-1} + 3^{k-1} = 3^k.$$

Finally, for $p = 2$ and $i > 0$, we have $\phi(2^i) = 2^{i-1}$, so

$$\Phi(2^k) = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1 = 2^k - 1.$$ ■

Corollary 2. *A prime power p^k is a PTN if and only if $p = 3$.*

Theorem 3. *If n is a PTN and $4n + 1$ is prime, then $3(4n + 1)$ is also a PTN.*

Proof. Let $m = 3(4n + 1)$ and $r = R(n)$. Since n is a PTN, we know that it is odd and $\phi^j(n)$ is even for $1 \leq j \leq r - 1$.

By the multiplicative property of ϕ, $\phi(m) = \phi(3(4n+1)) = \phi(3)\phi(4n+1)$. Since 3 and $4n+1$ are both prime, it follows that $\phi(3)\phi(4n+1) = 8n$. So $\phi(m) = 8n$. Again by the multiplicative property and the fact that m is odd, $\phi^2(m) = \phi(8n) = \phi(8)\phi(n) = 4\phi(n)$. For any even m, $\phi(4m) = 4\phi(m)$, so $\phi^k(4m) = 4\phi^k(m)$ as long as $\phi^{k-1}(4m)$ is even. Since $\phi(n)$ is even, for $3 \le i \le r+1$, we have $\phi^i(m) = \phi^{i-2}(\phi^2(m)) = \phi^{i-2}(4\phi(n)) = 4\phi^{i-1}(n)$. Also, $\phi^{r+2}(m) = \phi(\phi^{r+1}(m)) = \phi(4) = 2$, and similarly $\phi^{r+3}(m) = 1$. Now we are ready to find $\Phi(m)$.

$$\Phi(m) = \sum_{i=1}^{r+3} \phi^i(m) = \phi(m) + \sum_{i=2}^{r+1} \phi^i(m) + \phi^{r+2}(m) + \phi^{r+3}(m)$$

$$= 8n + \sum_{i=1}^{r} 4\phi(n) + 2 + 1 = 8n + 4\sum_{i=1}^{r} \phi(n) + 3.$$

Since n is a PTN, $n = \sum_{i=1}^{r} \phi(n)$ and it follows that

$$8n + 4\sum_{i=1}^{r} \phi(n) + 3 = 8n + 4n + 3 = 12n + 3 = 3(4n+1) = m. \qquad \blacksquare$$

Now for any PTN n, we check to see if $4n+1$ is prime. If so, then $3(4n+1)$ is also a PTN. By Theorem 2, powers of 3 are always PTNs, and we get the following corollary.

Corollary 3. *If* $4 \cdot 3^i + 1$ *is prime, then* $3(4 \cdot 3^i + 1)$ *is a PTN.*

It is not known whether there are infinitely many primes of the form $4 \cdot 3^i + 1$. Sequence A005537 of [**10**] gives the first eighteen such i as 0, 1, 2, 3, 6, 14, 15, 39, 201, 249, 885, 1005, 1254, 1635, 3306, 3522, 9602, 19785. The corresponding PTNs are 15, 39, 111, 327, and 8751, plus thirteen more that are greater than 100000; the largest of these has 9440 decimal digits.

Given these PTNs, we again check to see if they lead to more via Theorem 3. For example, since 39 is a PTN and $4 \cdot 39 + 1 = 157$ is prime, $3 \cdot 157 = 471$ is a PTN. However, $4 \cdot 471 + 1$ is not prime, and none of the other numbers listed give us more PTNs.

The next theorem involves Fermat primes, those of the form $2^{2^n} + 1$; its proof is straightforward.

Theorem 4. *If* $2^k - 1$ *is a PTN and* $2^k + 1$ *is prime, then* $(2^k - 1)(2^k + 1) = 2^{(2k)} - 1$ *is a PTN.*

This theorem gives us the PTNs $3 = 2^2 - 1$, $15 = 2^4 - 1$, $255 = 2^8 - 1$, $65535 = 2^{16} - 1$, and $4294967295 = 2^{32} - 1$. However $2^{32} + 1$ is not prime, and the chain terminates. It seems unlikely that there are any other PTNs of this type.

However, a return to Theorem 3 produces more PTNs. Starting with 15, we get 183 and 2199, but $4 \cdot 2199 + 1$ is not prime, and this chain ends. Similarly, 255 leads to the PTNs 3063 and 36759. The two largest PTNs above do not lead to more PTNs.

The following theorem is similar in nature, and the proof is a straightforward but involves substantial algebra and application of the above lemmas and the multiplicative property of ϕ. The hypotheses are very demanding, and we seem to obtain only a few more PTNs from them.

Theorem 5.

(a) *If both* $4(16(3^j)+1)+1$ *and* $16(3^j)+1$ *are prime, then* $27(4(16(3^j)+1)+1)$ *is a PTN.*

(b) *If both* $18(32(3^j)+1)+1$ *and* $32(3^j)+1$ *are prime, then* $9(18(32(3^j)+1)+1)$ *is a PTN.*

(c) *If all of* $6(6(16(3^j)+1)+1)+1$, $6(16(3^j)+1)+1$, *and* $16(3^j)+1$ *are prime, then* $9(6(6(16(3^j)+1)+1)+1)$ *is a PTN.*

In (a), the hypotheses are satisfied for $j = 3$, 4, and 12, giving us the PTNs 46791, 140103, and 918330183. Up to $j = 100$ these are the only values that satisfy the necessary criteria. We check these 3 examples with Theorem 3, but none satisfy that $4n + 1$ is prime.

Parts (b) and (c) give us one PTN each: 15723 ($j = 1$) and 5571 ($j = 0$) respectively, for $j \leq 100$. We can check these numbers with Theorem 3, but neither satisfies $4n + 1$ prime.

The following table summarizes our list of PTNs when this paper was first submitted. A computer search for PTNs between 100000 and 200000 turned up only two, both of them accounted for: 140103, listed below, and 177147, or 3^{11}.

Table 2. A Summary of Perfect Totient Numbers

PTN	Explanation
3^i	Theorem 2
363	Computer Search
4375	Computer Search
15, 39, 111, 327, 8751, 57395631, 172186887, 11 more	Corollary 2
183, 471, 2199, 3063, 4359, 36759	Theorem 3
15, 255, 65535, 4294967295	Theorem 4
46791, 140103, 918330183	Theorem 5(a)
15723	Theorem 5(b)
5571	Theorem 5(c)

To this list should be added the following seven PTNs, found by a computer search in [**2**]. The list now includes all PTNs less than $5 \cdot 10^9$. None of these are members of any known families of PTNs. They are as follows: $208191 = 3 \cdot 29 \cdot 239$, $441027 = 3^2 \cdot 49003$, $4190263 = 7 \cdot 11 \cdot 54419$, $9056583 = 3^3 \cdot 335429$, $236923383 = 3 \cdot 1427 \cdot 55343$, $3932935775 = 5^2 \cdot 29 \cdot 5424739$, and $4764161215 = 5 \cdot 11 \cdot 8662113$. We conclude this section with several questions regarding the existence of PTNs.

Question 1. *Is* 363 *the only PTN of the form* $3p^2$*? Is* 4375 *the only PTN whose only prime divisors are* 5 *and* 7*?*

In addition to 4375, Ianucci, Moujie, and Cohen [**2**] found three PTNs not divisible by 3: 4190263, 3932935775, and 4764161215.

Question 2. *Are there infinitely many primes of the form* $4 \cdot 3^i + 1$*?*

A positive answer would give us a second infinite family of PTNs. As mentioned before, [**10**] has a list of i for which $4 \cdot 3^i + 1$ is prime, but whether this list terminates is still unknown.

Question 3. *Can one find any other PTNs, either by a computer search or by analytic means?*

Question 4. *What is the range of* Φ*?*

It is clear that $\Phi(n)$ is never even. In addition, [**5**] proves that if n is in the range of Φ, then $2n + 1$ is as well.

When this paper was first submitted, we believed the material was new. A referee's report referred us to [**4**], [**5**], [**6**], and [**8**], all written in Spanish between 1939 and 1958. Searches for papers referencing these led us to [**12**], [**2**], [**3**], and [**11**], and the term 'perfect totient number'. Thus little of the material other than on iterating Φ is new. Here we keep the development of the original paper, but consider the bulk of it an exposition, not a work of new mathematics. In the final section, we provide a survey of the literature on perfect totient numbers and the Φ function.

Iterating Φ

Catching the iteration bug, we can ask, "What if we iterate Φ?" When we do this, PTNs become fixed points and the corresponding structure is no longer connected.

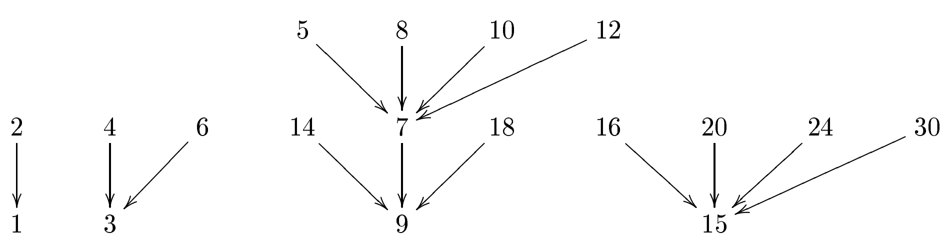

Figure 2. Φ trees

Question 5. *Given n, what happens to the sequence* $\Phi^i(n)$*? Do any such sequences diverge?*

For the first 106 natural numbers, the sequence $\Phi^i(n)$ eventually reaches a PTN. For example, $\Phi^i(40) = 31, 45, 39, 39, 39, \ldots$. The first n for which $\Phi^i(n)$ might diverge is 107; we know that $\Phi^{1000}(107) \approxeq 8.8 \times 10^{23}$. We know, though, that not all Φ^i sequences reach a PTN. For example, $\Phi(579) = 639$, while $\Phi(639) = 579$, creating a cycle of length 2. Other 2-cycles are formed by the pairs $\{14911, 18207\}$, $\{38575, 47223\}$, and $\{310399, 492855\}$. There are also 3-cycles: $\{20339, 23883, 21159\}$, $\{35503, 43255, 45375\}$, and $\{365399, 493047, 476343\}$. There are no other cycles with period less than 15 containing numbers less than 100000. We still ask whether there are other cycles and what are their lengths?

Pillai [**7**] and Shapiro [**9**] partition the natural numbers into classes according to $R(n)$, the number of iterations of ϕ required to reach 1. Here we classify the natural numbers according to their ultimate destination under iteration of Φ. Given a number n, let E_n be the set of all numbers k for which $(\Phi)^i(k) = n$ for some i. That is, $|E_n|$

counts how many numbers eventually reach n under iteration of Φ. We are most interested in E_n for n a PTN or part of a cycle. In agreement with the trees above, $|E_3| = 3$, $|E_9| = 8$, and $|E_{15}| = 5$. Also, $|E_1| = 2$, $|E_{27}| = 10$, $|E_{39}| = 27$, and $|E_{81}| = 2$. Using Shapiro's lower bound $\phi(n) > n^{(\log 2)/(\log 3)}$ as a (naive) lower bound for $\Phi(n)$, we can determine when all members of a given E_n have been found. That is, if e_n is the largest known member of E_n, the first larger member of E_n must be less than $e_n^{(\log 2)/(\log 3)}$.

Question 6. *Is E_n finite for all n?*

Literature summary

Though the term would not be coined until 1975, perfect totient numbers were first studied by Perez Cacho [**4**] in 1939. He proved our Theorems 2 and 4, as well as Theorem 3 and its converse: for an odd prime p, $3p$ is a PTN if and only if $p = 4x + 1$, with x a PTN. In [**5**], the same author showed:

- $\Phi(n) = 2n - 3$ if and only if n is a Fermat prime.
- The only numbers of the form $n = 2^k \pm 1$ for which $\Phi(n) \equiv 1 \pmod 4$ are the Mersenne primes.
- $\Phi(n) = n - 1$ if and only if $n = 2^k$.

In 1950, Rodeja [**8**] improved a condition from [**4**], proving that $3(4 \cdot 3^p + 1)$ is the only PTN of the form $(2^k + 1)^m (2^l (2^k + 1)^p + 1)^s$. In [**11**], Subbarao, unaware of the previous work in Spanish, proved that the powers of 3 and numbers obtained via Theorem 3 are PTNs and conjectured that these are the only PTNs. This is the first appearance of the notion of Φ or a PTN in English. Venkataraman [**12**] coined the term *perfect totient number*, but was also unaware of the work of Perez Cacho and Rodeja. He proved Corollary 2 and that $4 \cdot 3^i + 1$ is never of the form k^t for $k, t > 1$. Mohan and Suryanarayana [**3**] proved Theorems 5 and 6, and proved that $3p$ is never a PTN if $p \equiv 3 \bmod 4$. They also further characterized PTNs involving Fermat primes. Mohan and Suryanarayana appear to be the first authors to be aware of all previous results on the subject in both English and Spanish.[1]

In a paper that appeared after our original submission of this paper, Iannucci, Moujie, and Cohen [**2**] proved Theorem 7 and three other sufficient conditions for $3^2 p$ to be a PTN. They also gave two sufficient conditions that $3^3 p$ be a PTN and ruled out certain PTNs of the form $3^k p$ with $k \geq 4$.

References

1. D. M. Burton, *Elementary Number Theory*, 5th ed., McGraw-Hill, 2002.
2. D. Iannucci, D. Moujie, and G. Cohen, On Perfect Totient Numbers, *J. Integer Seq.* **6** (2003) Article 03.4.5.
3. A. L. Mohan and D. Suryanarayana, Perfect Totient Numbers, In *Number Theory (Proc. Third Mathematics Conf., Mysore, 1981)*, Lecture Notes in Mathematics 938, Springer-Verlag, (1982) 101–105.
4. L. Perez Cacho, Sobre la Suma de Indicadores de Ordenes sucesivos, *Revista Matematica Hispano-Americana* **1** (1939) 45–50.

[1] The authors take some consolation in finding themselves not alone in being unaware of previous results on PTNs.

5. L. Perez Cacho, Funcion Suma de Indicadores sucesivos, *Memorias de Matematica del Instituto "Jorge Juan"* **7** (1948) 5–24.
6. L. Perez Cacho, Sobre algunas Cuestiones de la Teoria de Numeros, *Revista Matematica Hispano-Americana* **18** (1958) 10–27, 113–124.
7. S. Sivasankaranarayana Pillai, On a Function connected with $\phi(n)$, *Bull. Amer. Math. Soc.* **35** (1929) 837–841.
8. E. G.-Rodeja F., Sobre una Cuestion de Funciones de Euler de Ordenes Sucesivos, *Revista Matematica Hispano-Americana* **10** (1950) 95–98.
9. H. Shapiro, An Arithmetic Function arising from the ϕ function, *Amer. Math. Monthly* **50** (1943) 18–30.
10. N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. `http://www.research.att.com/~njas/sequences`, sequence A005537.
11. M. V. Subbarao, On a Function connected with $\phi(n)$, *Math. Student* **23** (1955) 178–179.
12. T. Venkataraman, Perfect Totient Number, *Math. Student* **43** (1975) 178.
13. H. C. Williams and C. R. Zarnke, Some prime numbers of the forms $2A3^n + 1$ and $2A3^n - 1$, *Math. Comp.* **26** (1972) 995–998.

Teaching Tip: An Integration Technique

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When my class was faced with $\int \sqrt{\cosh^2 x - \cosh x}\, dx$ (which arose in an arc length problem), this question came up: How does one integrate rational functions of hyperbolic sines and cosines? In other words, if $P(x, y)$ and $Q(x, y)$ are polynomials in two variables, is there a technique for finding

$$\int \frac{P(\cosh t, \sinh t)}{Q(\cosh t, \sinh t)}\, dt?$$

Although our result is almost certainly not new, we have not found it anywhere. What we do is to emulate the ordinary (circular) trig technique: we let $u = \tanh \frac{t}{2}$. Then, using the identities

$$\cosh 2t = \cosh^2 t + \sinh^2 t \quad \text{and} \quad \sinh 2t = 2 \sinh t \cosh t,$$

we get

$$\cosh\big(2 \tanh^{-1} u\big) = \frac{1 + u^2}{1 - u^2} \quad \text{and} \quad \sinh\big(2 \tanh^{-1} u\big) = \frac{2u}{1 - u^2}.$$

Since $\frac{d}{du} \tanh u = \frac{1}{1-u^2}$, after substitution the above integral becomes

$$\int \frac{P\left(\frac{1+u^2}{1-u^2}, \frac{2u}{1-u^2}\right)}{Q\left(\frac{1+u^2}{1-u^2}, \frac{2u}{1-u^2}\right)} \frac{1}{1 - u^2}\, du.$$

This integrand is just a rational function of u, and so the integral can be attacked with the usual weapons.

