Paley type partial difference sets in non p-groups

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Abstract

By modifying a construction for Hadamard (Menon) difference sets we construct two infinite families of negative Latin square type partial difference sets in groups of the form ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_p}^{4t}$ where p is any odd prime. One of these families has the well-known Paley parameters, which had previously only been constructed in p-groups. This provides new constructions of Hadamard matrices and implies the existence of many new strongly regular graphs including some that are conference graphs. As a corollary, we are able to construct Paley-Hadamard difference sets of the Stanton-Sprott family in groups of the form ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_p}^{4t} \times \mathbb{Z}_{9p^{4t}\pm 2}$ when $9p^{4t} \pm 2$ is a prime power. These are new parameters for such difference sets.

1 Introduction

Let G be a finite group of order v, and let D be a subset of G with cardinality k. Then D is a (v, k, λ)-*difference set* (DS) if the list of differences $d_1 d_2^{-1}$, $d_1, d_2 \in D$ represents every nonidentity element in G exactly λ times. A difference set, D, is called *reversible* if $d \in D$ implies $d^{-1} \in D$. *Hadamard (Menon) difference sets*, having parameters $(4u^2, 2u^2 - u, u^2 - u)$, are of particular interest due to the fact that their ± 1 incidence matrices form Hadamard matrices. The text of Beth, Jungnickel, and Lenz [1] and the survey of Jungnickel [7] are excellent references for DSs.

Similarly, suppose G is a finite group of order v with a subset D of order k such that the differences ${d_1 d_2}^{-1}$ for $d_1, d_2 \in D, d_1 \neq d_2$ represent each nonidentity element of D exactly λ times and the nonidentity element of $G - D$ exactly μ times. Then D is called a (v, k, λ, μ)-*partial difference set (PDS)* in G. When the identity $e \notin D$ and $D^{(-1)} = D$ we call the PDS D *regular*. The survey article of Ma is a very good survey on PDSs [9]. A partial difference set having parameters $(n^2, r(n+1), -n + r^2 + 3r, r^2 + r)$ is called a *negative Latin square type PDS*.

Another important family of difference sets is the *Paley-Hadamard difference sets* having parameters $(v, \frac{v-1}{2}, \frac{v-3}{4})$. When such a difference set also has the property that G is the disjoint union of D, the inverses of the elements of D, and 0, it is called a *skew Hadamard difference set*. Closely related to Paley-Hadamard difference sets are *Paley type partial difference sets*, having parameters $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$. Paley originally discovered these sets [10] along with the Paley-Hadamard difference sets in the context of Hadamard matrices, and in fact both Paley PDSs and Paley-Hadamard difference sets can be used to construct both Hadamard designs and matrices. In the survey of Ma [9] were the following questions:

"**Questions 13.4.** Suppose G is an abelian group of order $v \equiv 1 (\mathrm{mod}\ 4)$. If v is not a prime power, does there exist a Paley PDS in G? If v is a prime power, does G need to be elementary abelian?"

The articles of Davis as well as Leung and Ma give various constructions of Paley partial difference sets in p-groups ([3], [8]) that are not elementary abelian and thus answer the second question. In this paper, we will give the first construction of Paley PDSs in groups having an order which is not a prime power and therefore provide the first positive answer to the first of the questions. Using these Paley PDSs, we will also construct Paley-Hadamard DSs with new parameters.

There have also been relatively few constructions of negative Latin square type partial difference sets. Originally most of these were in nonelementary abelian groups, [9]. Recently, there have been constructions given in nonelementary abelian p-groups such as Davis and Xiang in [5] and Polhill [11]. Jørgensen and Klin [6] constructed negative Latin square type PDSs in groups of order 100. In this paper, we will not only construct the Paley PDSs but another pair of families of negative Latin square type PDSs in non p-groups as well.

Regular partial difference sets generate Cayley graphs which are strongly regular, and in particular the Paley-type and negative Latin square type PDSs give rise to conference graphs and negative Latin square type strongly regular graphs respectively.

Often DSs and PDSs are studied within the context of the group ring $\mathbf{Z}[\mathbf{G}]$. For a subset D in G we can write $D = \sum_{d \in D} d$ and $D^{(-1)} = \sum_{d \in D} d^{-1}$. This is abuse of notation which is widely accepted; so that depending on the context D will represent the difference set D or the element $\sum_{d \in D} d$ in the group ring $\mathbb{Z}[G]$. Character theory is frequently used to simplify calculations with difference sets and partial difference sets in abelian groups. Turyn [13] and separately Yamamoto [18] first used character theory to study abelian difference sets. A *character* on an abelian group G is a homomorphism from the group to the complex numbers with modulus 1. The *principal character* is 1 on all the elements of G; any other character is called *nonprincipal*. One can naturally extend a character on G to a homomorphism of the group ring $\mathbb{Z}[G]$ as follows: if χ is a character on G, then for an element $A = \sum_{g \in G} a_g g$ let $\chi(A) = \sum_{g \in G} a_g \chi(g)$ so that if S is a subset of G, then $\chi(S) = \sum_{s \in S} \chi(s)$. See [13] for a proof of similar results to the following.

Theorem 1.1 *(a) Let G be an abelian group of order v with a subset D of cardinality k and let λ be a positive integer satisfying $\lambda(v-1) = k(k-1)$. Then D is a (v, k, λ)-difference set in G if and only if for every nonprincipal character χ on G, $|\chi(D)| = \sqrt{k - \lambda}$.*

(b) Let G be an abelian group of order v with a subset P of cardinality k such that $0 \notin P$ with $k^2 = k + \lambda k + \mu(v - k - 1)$ for positive integers λ and μ. Then P is a (v, k, λ, μ)-partial difference set in G if and only if for every nonprincipal character χ on G, $\chi(P) = \frac{(\lambda-\mu) \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2}$.

2 Known Constructions of Hadamard Difference Sets

Of particular relevance to this paper is the Hadamard difference set construction for $K \times {\mathbb{Z}_p}^{4t}$ where $|K| = 4$ and p is an odd prime. Xia first constructed such difference sets for $p \equiv 3 \bmod 4$ in [16], and a simplified construction was given by Xiang and Chen [17] using the additive characters of finite fields. Then Wilson and Xiang [15] related the existence of this type of difference sets to certain projective sets, and finally Chen used their work to construct Hadamard difference sets in all groups $K \times {\mathbb{Z}_p}^{4t}$ for $|K| = 4$ and p an odd prime [2].

Let $PG(k-1,q)$ denote the projective space of dimension $k-1$ over F_q for q a power of a prime; the corresponding vector space $V(k,q)$ for $PG(k-1,q)$ will have dimension k over F_q. The elements of $PG(k-1,q)$ are the subspaces of $V(k,q)$, and, in particular, a projective point is a 1-dimensional subspace, a projective line is a 2-dimensional space, and a hyperplane is a $(k-1)$-dimensional subspace. A *projective* (n,k,h_1,h_2) *set* $\mathcal{O}$ is a proper subset of the projective space $PG(k-1,q)$ with n points ($n \neq 0$) so that $\mathcal{O}$ intersects every hyperplane in h_1 or h_2 points. If we have that $\mathcal{O} = \{\langle y_1\rangle, \langle y_2\rangle, ..., \langle y_n\rangle\}$ then let $\Omega = \{x \in V(k,q) | \langle x\rangle \in \mathcal{O}\}$. The following lemma shows that $\mathcal{O}$ is a projective set if and only if Ω is a PDS in the additive group of the corresponding vector space under certain conditions. Wilson and Xiang have this result in [15].

Lemma 2.1 *$\mathcal{O}$ is a projective (n,k,h_1,h_2) set if and only if $\chi(\Omega) = qh_1 - n$ or $qh_2 - n$ for every nonprincipal additive character.*

Using this lemma we see that $\mathcal{O}$ is a projective (n,k,h_1,h_2) set if and only if Ω is a PDS in the additive group of $V(k,q)$ provided that $K^2 = K + \lambda K + \mu(v - K - 1)$ (K is the cardinality of Ω, while λ and μ are the usual PDS parameters).

Now consider the projective space $\Sigma_3 = PG(3,q)$ for an odd prime power $q = p^t$. The additive group of the corresponding vector space is ${\mathbb{Z}_p}^{4t}$. Define a *spread* of Σ_3 to be a set of q^2+1 projective lines which are pairwise disjoint and partition the points of Σ_3. Finally, a subset of *Type Q* is a projective $(\frac{q^4-1}{4(q-1)}, 4, \frac{(q-1)^2}{4}, \frac{(q+1)^2}{4})$ set in Σ_3.

Theorem 2.2 (Xiang-Wilson [15]) *Suppose that $S = \{L_1, L_2, ..., L_{q^2+1}\}$ is a spread of Σ_3. If there exist two subsets of Type Q, C_0 and C_1, in Σ_3 with the property that $|C_0 \cap L_i| = (q+1)/2$ for $1 \leq i \leq \frac{q^2+1}{2}$ and $|C_1 \cap L_j| = (q+1)/2$ for $\frac{q^2+1}{2} + 1 \leq j \leq q^2+1$, then there exists a Hadamard difference set in $K \times {\mathbb{Z}_p}^{4t}$ for $|K| = 4$.*

We make some observations regarding this Theorem. If two such subsets of Type Q exist, then the sets

$$C_2 = \cup_{1 \leq i \leq \frac{q^2+1}{2}} L_i \setminus C_0 \text{ and}$$

$$C_3 = \cup_{\frac{q^2+1}{2}+1 \leq j \leq q^2+1} L_j \setminus C_1$$

are also subsets of Type Q.

Now let $W = V(k,q)$ be the underlying vector space of Σ_3. Then let $\mathcal{C}_i = \{w \in W | \langle w\rangle \in C_i\}$ and $\mathcal{L}_i = \{w \in W | \langle w\rangle \in L_i\}$.

Each of the four subsets C_i in Σ_3 gives rise to a PDS $\mathcal{C}_i$ in $V(4,q)$ such that the character values of each are either $\frac{q^2-1}{4}$ or $\frac{q^2-1}{4} - q^2$ for nonprincipal additive characters on $V(4,q)$. The character sum will be $\frac{q^2-1}{4} - q^2$ for exactly one of the sets $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ for any nonprincipal character.

To each of $\mathcal{C}_0$ and $\mathcal{C}_2$ we will add $\frac{q^2-1}{4}$ of the sets $\mathcal{L}_j$ which are disjoint from $\mathcal{C}_0$ and $\mathcal{C}_2$; we call this collection of $\frac{q^2-1}{4}$ projective lines B_1 with corresponding set $\mathcal{B}_1 = \{w \in W | \langle w \rangle \in B_1\}$. The character sum on $\mathcal{B}_1$ will be either $-\frac{q^2-1}{4}$ or $q^2 - \frac{q^2-1}{4}$, and when combined with the PDSs from subsets of Type Q, the character sums are either 0 or $\pm q^2$.

We similarly add $\frac{q^2-1}{4}$ of the sets $\mathcal{L}_i$ (disjoint from $\mathcal{C}_1$ and $\mathcal{C}_3$) to $\mathcal{C}_1$ and $\mathcal{C}_3$, and call this collection B_0. When we take the four sets

$$D_0 = \mathcal{C}_0 \cup \mathcal{B}_1, \quad D_1 = \mathcal{C}_1 \cup \mathcal{B}_0, \quad D_2 = \mathcal{C}_2 \cup \mathcal{B}_1, \quad D_3 = \mathcal{C}_3 \cup \mathcal{B}_0,$$

the character sums will be such that for any nonprincipal character χ on $G = \mathbb{Z}_p{}^{4t}$, $\chi(D_i) = \pm q^2$ for exactly one i and $\chi(D_k) = 0$ for all other k. Then the four sets $G \setminus D_0, D_1, D_2, D_3$ form a $(\frac{p^4-p^2}{2}, p^2, 4, +)$ covering extended building set using the terminology from [4], so that if $K = \{a_0, a_1, a_2, a_3\}$ then

$$H = a_0(G \setminus D_0) \cup a_1 D_1 \cup a_2 D_2 \cup a_3 D_3$$

is a Hadamard difference set in $K \times \mathbb{Z}_p{}^{4t}$.

We form two more sets B_2 and B_3 as follows: $\mathcal{B}_2 = (\mathcal{C}_0 \cup \mathcal{C}_2) \setminus \mathcal{B}_0$ and $\mathcal{B}_3 = (\mathcal{C}_1 \cup \mathcal{C}_3) \setminus \mathcal{B}_1$ so that each of B_2 and B_3 are unions of $\frac{q^2+3}{4}$ projective lines. The properties of character sums on the $\mathcal{B}_i$ are given in the following lemma.

Lemma 2.3 *Let χ be a nonprincipal character on $V(4,q)$. Then $\chi(\mathcal{B}_0), \chi(\mathcal{B}_1) \in \{-\frac{q^2-1}{4}, q^2 - \frac{q^2-1}{4}\}$ and $\chi(\mathcal{B}_2), \chi(\mathcal{B}_3) \in \{-\frac{q^2+3}{4}, q^2 - \frac{q^2+3}{4}\}$. Furthermore, exactly one $\mathcal{B}_i$ will have a positive character sum.*

Proof: The character sums immediately follow from Theorem 1.1. The fact that there will be exactly one positive character sum follows from the fact that $\Sigma_3 = B_0 \cup B_1 \cup B_2 \cup B_3$. $\square$

Now we wish to form two more sets L and M in the additive group of $V(4,q)$. $L = \mathcal{B}_1 \cup \mathcal{B}_2$. The set L, viewed projectively, is a union of $\frac{q^2+1}{2}$ projective lines, and in fact is a Paley-type PDS in the additive group of $V(4,q)$. The set $M = \mathcal{B}_2 \cup \mathcal{B}_3$, viewed projectively, is a union of $\frac{q^2+3}{2}$ lines and is also a PDS in the additive group of $V(4,q)$. These PDSs have special properties with respect to the sets D_i as given in the following key Lemmas.

Lemma 2.4 *Let χ be a nonprincipal character on $V(4,q)$. For $i = 0$ or 2, if $\chi(G \setminus D_i) = \pm q^2$ then $\chi(L) = \frac{\mp q^2-1}{2}$. For $i = 1$ or 3, if $\chi(D_i) = \pm q^2$ then $\chi(L) = \frac{\mp q^2-1}{2}$.*

Proof: There will be exactly one $\mathcal{L}_i$ on which χ is principal. Suppose $\chi(D_0) = q^2$ so $\chi(G \setminus D_0) = -q^2$. Then it must be the case that $\chi(\mathcal{B}_1)$ is positive. It follows that $\chi(L) = \frac{q^2-1}{2}$. If instead $\chi(D_0) = -q^2$ then $\chi(G \setminus D_0) = q^2$. It follows that $\chi(\mathcal{B}_1)$ is negative. Moreover, $\chi(\mathcal{C}_0) = \frac{q^2-1}{4} - q^2$ so that $\chi(\mathcal{C}_0 \cup \mathcal{C}_2) = \chi(\mathcal{B}_0 \cup \mathcal{B}_2) = \frac{-q^2-1}{2}$. This ensures that $\chi(\mathcal{B}_0)$ and $\chi(\mathcal{B}_2)$ must both be negative. Then it must be the case that $\chi(\mathcal{B}_3)$ is positive, so that $\chi(L) = \frac{-q^2-1}{2}$. The cases $i \neq 0$ are similar. $\square$

Lemma 2.5 *Let χ be a nonprincipal character on $V(4,q)$. If $\chi(D_i) = \pm q^2$ then $\chi(M) = \frac{\mp q^2-3}{2}$.*

3 Paley PDSs in Non p-Groups

Let $G = {\mathbb{Z}_p}^{4t}$. Using the same sets $D_i \subset G$ as in Xiang-Wilson's Theorem above, we can replace the group K with ${\mathbb{Z}_3}^2$ and will be able to form negative Latin square type PDSs in any group of the form $G' = {\mathbb{Z}_3}^2 \times G$. Let H_0, H_1, H_2, and H_3 be the subgroups of order 3 in ${\mathbb{Z}_3}^2$. ${H_i}^* = H_i \setminus \{(0,0)\}$ so that ${H_i}^* \cap {H_j}^* = \emptyset$ for $i \neq j$. This will ensure that the set D given in Theorem 3.1 is not a multiset.

Theorem 3.1 (Paley partial difference sets in non p-groups) *Let $G' = {\mathbb{Z}_3}^2 \times G$ where $G = {\mathbb{Z}_p}^{4t}$ for an odd prime p. Then the set $D = (\{(0,0)\} \times L) \cup ({H_0}^* \times (G \setminus D_0)) \cup ({H_1}^* \times D_1) \cup ({H_2}^* \times (G \setminus D_2)) \cup ({H_3}^* \times D_3)$ is a $(9p^{4t}, \frac{9p^{4t}-1}{2}, \frac{9p^{4t}-5}{4}, \frac{9p^{4t}-1}{4})$ Paley-type PDS in G'.*

Proof: First we check to see that D has the correct cardinality.

$$|D| = |L| + 2|G \setminus D_0| + 2|D_1| + 2|G \setminus D_2| + 2|D_3| = \frac{p^{4t}-1}{2} + 4|G| = \frac{p^{4t}-1}{2} + 4p^{4t} = \frac{|G'|-1}{2}.$$

Suppose that ϕ is a nonprincipal character on $G' = {\mathbb{Z}_3}^2 \times G$. Then $\phi = \chi \otimes \psi$, where χ is a character on ${\mathbb{Z}_3}^2$ and ψ is a character on G. To use Theorem 1.1, we need to show that $\phi(D) = \frac{\pm 3p^{2t}-1}{2}$.

Case 1: χ is principal on ${\mathbb{Z}_3}^2$, but ψ is nonprincipal on G. Then $\chi({H_i}^*) = 2$ for all i. $\psi(D_j) = \pm q^2 = \pm p^{2t}$ for exactly one j, and $\psi(D_i) = 0$ for $i \neq j$. If $j = 1$ or 3 then by Lemma 2.4, if $\psi(D_j) = p^{2t}$ then $\psi(L) = \frac{-p^{2t}-1}{2}$ while if $\psi(D_j) = -p^{2t}$ then $\psi(L) = \frac{p^{2t}-1}{2}$. We will have then:

$$\phi(D) = \frac{\pm p^{2t}-1}{2} + 2(\mp p^{2t}) = \frac{\mp 3p^{2t}-1}{2}.$$

A similar argument works for $j = 0$ or 2.

Case 2: χ is nonprincipal on ${\mathbb{Z}_3}^2$, but ψ is principal on G. Then $\psi(L) = \frac{p^{4t}-1}{2}$, $\psi(G \setminus D_i) = |G \setminus D_i| = \frac{p^{4t}+p^{2t}}{2}$, and $\psi(D_i) = |D_i| = \frac{p^{4t}-p^{2t}}{2}$. χ will be principal on exactly one of the H_j so that $\chi(H_j^*) = 2$ while $\chi(H_i^*) = -1$ for $i \neq j$. If $j = 0$ or 2 we get:

$$\chi(D) = |L| + (2-1)(\frac{p^{4t}+p^{2t}}{2}) + (-1-1)(\frac{p^{4t}-p^{2t}}{2}) = \frac{3p^{2t}-1}{2}.$$

If $j = 1$ or 3 we get:

$$\phi(D) = |L| + (-1-1)(\frac{p^{4t}+p^{2t}}{2}) + (2-1)(\frac{p^{4t}-p^{2t}}{2}) = \frac{-3p^{2t}-1}{2}.$$

Case 3: Suppose that both χ and ψ are nonprincipal. Then χ will be principal on exactly one of the H_k so that $\chi(H_k^*) = 2$ while $\chi(H_i^*) = -1$ for $i \neq k$. $\psi(D_j) = \pm q^2 = \pm p^{2t}$ for exactly one j, and $\psi(D_i) = 0$ for $i \neq j$. If $j = 0$ or 2 then by Lemma 2.4, if $\psi(G \setminus D_j) = p^{2t}$ then $\psi(L) = \frac{-p^{2t}-1}{2}$ while if $\psi(G \setminus D_j) = -p^{2t}$ then $\psi(L) = \frac{p^{2t}-1}{2}$. If $k = j$ we get:

$$\phi(D) = \frac{\pm p^{2t}-1}{2} + 2(\mp p^{2t}) = \frac{\mp 3p^{2t}-1}{2}.$$

If $k \neq j$ then we get:

$$\phi(D) = \frac{\pm p^{2t}-1}{2} - 1(\mp p^{2t}) = \frac{\pm 3p^{2t}-1}{2}.$$

A similar argument works for $j = 1$ or 3. $\square$

We have hence provided a method to construct Paley PDSs in certain groups having an order that is not a prime power. The following theorem gives another family of negative Latin square type PDSs that will generate a family of negative Latin square type graphs with number of vertices that is not a prime power. We omit the proof since it is extremely similar to the previous.

Theorem 3.2 *Let $G' = {\mathbb{Z}_3}^2 \times G$ where $G = {\mathbb{Z}_p}^{4t}$ for an odd prime p. Then the set $S = (\{(0,0)\} \times M) \cup ({H_0}^* \times D_0) \cup ({H_1}^* \times D_1) \cup ({H_2}^* \times D_2) \cup ({H_3}^* \times D_3)$ is a $(9p^{4t}, r(3p^{2t}+1), -3p^{2t}+r^2+3r, r^2+r)$-negative Latin square type PDS in G' for $r = \frac{3p^{2t}-3}{2}$.*

We can also get another set of parameters for negative Latin square type partial difference sets by taking $T = (G' \times G)^* \setminus S$. As an example, in the group ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_5}^4$ we obtain from Theorem 3.1 a $(75^2, 37(75+1), -75+37^2+3(37), 37^2+37)$-Paley PDS which we call D. From Theorem 3.2 we obtain PDSs S and T which are $(75^2, 36(75+1), -75+36^2+3(36), 36^2+36)$- and $(75^2, 38(75+1), -75+38^2+3(38), 38^2+38)$-negative Latin square type partial difference sets. Thus we have three strongly regular graphs D, S, and T of the negative Latin square type on $75^2 = 5625$ vertices, having degrees 2812, 2736, and 2888 respectively. The Cayley graph from D will in particular be a conference graph.

4 New Paley-Hadamard Difference Sets

We will now show how the Paley PDSs constructed in Theorem 3.1 can be used to construct new Paley-Hadamard difference sets. The following is due to Stanton and Sprott [12].

Theorem 4.1 *Suppose that q and $q+2$ are both prime powers. Then there is a $(q(q+2), \frac{q(q+2)-1}{2}, \frac{q(q+2)-3}{2})$-Hadamard difference set in $EA(q) \times EA(q+2)$, where $EA(q)$ denotes the elementary abelian group or order q and is the additive group of F_q.*

In [14], there are the following generalizations of Stanton and Sprott's earlier result.

Theorem 4.2 *Suppose that the group G has a $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$-Paley partial difference set P with $0 \notin P$ and that the group G' having order $v+2$ has a $(v+2, \frac{v+1}{2}, \frac{v-1}{4})$-skew Hadamard difference set S. Then $D = (G \times \{0\}) \cup (P \times S) \cup ((G^* \setminus P) \times (G'^* \setminus S))$ is a $(v(v+2), \frac{v(v+2)-1}{2}, \frac{v(v+2)-3}{4})$- Paley-Hadamard difference set in the group $G \times G'$.*

Theorem 4.3 *Suppose that the group G has a $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$-Paley partial difference set P with $0 \notin P$ and that the group having order $v-2$ has a $(v-2, \frac{v-3}{2}, \frac{v-5}{4})$-skew Hadamard difference set S. Then $D = (\{0\} \times G') \cup (P \times S) \cup ((G^* \setminus P) \times (G'^* \setminus S))$ is a $(v(v-2), \frac{v(v-2)-1}{2}, \frac{v(v-2)-3}{4})$- Paley-Hadamard difference set in the group $G \times G'$.*

Combining Theorem 3.1 with these two results we can get new Stanton-Sprott difference sets. These difference sets will have parameters distinct from previous constructions. A few examples of such Paley-Hadamard DSs are in the following groups: ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_5}^4 \times \mathbb{Z}_{5623}$, ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_5}^8 \times \mathbb{Z}_{3515623}$, ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_5}^{12} \times \mathbb{Z}_{2197265627}$, ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_7}^4 \times \mathbb{Z}_{21611}$, and ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_7}^8 \times \mathbb{Z}_{51883211}$. In each case, we have a Paley PDS from Theorem 3.1 in the first two components and a skew Hadamard difference set in the latter component since the latter is $\mathbb{Z}_p$ for a prime $p \equiv 3 (\text{mod } 4)$. We summarize with the following corollary.

Corollary 4.4 *Suppose that for an odd prime p that $9p^{4t} \pm 2$ is a prime power. Then there is a Paley-Hadamard difference set in ${\mathbb{Z}_3}^2 \times {\mathbb{Z}_p}^{4t} \times \mathbb{Z}_{9p^{4t} \pm 2}$.*

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