

INTRINSIC GEOMETRY OF A EUCLIDEAN SIMPLEX

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ABSTRACT. We give a simple technique to compute the distance between two points in an n -dimensional Euclidean simplex, where the points are given in barycentric coordinates, using only the edge lengths of that simplex. We then use this technique to verify a few computations which will be used in subsequent papers. The most important application is a formula for intrinsically computing the volume of a Euclidean simplex which is more efficient (and more natural) than any previously documented methods.

1. INTRODUCTION

While studying the isometric embedding problem for metric simplicial complexes in [Min15], the author came across the following basic problem. In attempting to work out basic examples, one needs to be able to compute distances between points in a given Euclidean simplex given only the barycentric coordinates of those points and the lengths of the edges of that simplex. More specifically, let $\sigma = \langle v_0, v_1, \dots, v_n \rangle$ be an (abstract) n -dimensional simplex with vertices v_0, \dots, v_n , let e_{ij} denote the edge connecting the vertices v_i and v_j , let γ_{ij} denote the length assigned to e_{ij} , and let $x, y \in \sigma$ with barycentric coordinates $x = (x_0, \dots, x_n)$ and $y = (y_0, \dots, y_n)$. The questions that needed to be answered were:

- (1) For what values of γ_{ij} can σ be realized as a legitimate simplex in \mathbb{E}^n .
- (2) Assuming that we have “good” values γ_{ij} , give a simple formula to calculate $d_\sigma(x, y)$, the length of the straight line segment connecting x to y within σ .

A naive attempt to solve question (2) is to construct an explicit isometric embedding of σ into \mathbb{E}^n , and then compute $d_\sigma(x, y)$ using basic Euclidean geometry. But for $n \geq 3$ constructing this embedding becomes quite cumbersome, and a much simpler method is described in Section 2.

Question (1) is an old problem, and was first solved by Cayley in [Cay41] way back in 1841. Question (1) was also solved for hyperbolic and spherical simplices by Karliža in [Kar04]. Another solution to question (1), as well as the main ingredient to the solution to question (2), can be found in an arXiv paper¹ by Igor Rivin [Riv03]. But Rivin does not use his formula for the Gram matrix to show how to compute distances interior to a simplex, which is the main issue that we take up in this paper. In Sections 3, 4, and 5 we demonstrate the power of formula (2.3) by working out some computations which would be difficult to produce directly. Most notably though is Theorem 4 which, in conjunction with equation (2.3), gives a simple formula for intrinsically computing the volume of any given Euclidean n -simplex. This formula is computationally simpler than the widely used *Cayley-Menger determinant*, as will be discussed in Section 6.

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¹While researching this problem I completely missed this paper. I have only found this paper very recently.

2. THE MAIN FORMULA

Linearly embed the n -simplex σ into \mathbb{R}^n in some way, and by abuse of notation we will identify each vertex v_i with its image in \mathbb{R}^n . For each i let $w_i := v_i - v_0$, so w_i is just the vector in \mathbb{R}^n representing the edge e_{0i} . Since σ is embedded in \mathbb{R}^n , the collection $\{w_i\}_{i=1}^n$ forms a basis for \mathbb{R}^n . If we had values for $\langle w_i, w_j \rangle$ then we could use those values to define a symmetric bilinear form on \mathbb{R}^n . But observe that due to the symmetry and bilinearity of $\langle \cdot, \cdot \rangle$:

$$\langle w_i - w_j, w_i - w_j \rangle = \langle w_i, w_i \rangle - 2\langle w_i, w_j \rangle + \langle w_j, w_j \rangle$$

and so

$$(2.1) \quad \langle w_i, w_j \rangle = \frac{1}{2} (\langle w_i, w_i \rangle + \langle w_j, w_j \rangle - \langle w_i - w_j, w_i - w_j \rangle).$$

Now, if our embedding of σ into \mathbb{R}^n were an isometry, then for all i we would have $\langle w_i, w_i \rangle = \gamma_{0i}^2$. Equation (2.1) would then become

$$(2.2) \quad \langle w_i, w_j \rangle = \frac{1}{2} (\gamma_{0i}^2 + \gamma_{0j}^2 - \gamma_{ij}^2).$$

where $\gamma_{ij}^2 := 0$ if $i = j$.

The trick now is to *define* the symmetric bilinear form $\langle \cdot, \cdot \rangle$ by equation (2.2). This naturally defines a quadratic form Q on \mathbb{R}^n (using only the edge lengths assigned to σ , and our original choice of embedding). It is easy to see that an orthogonal transformation will map our original embedding to an isometric embedding with respect to the standard Euclidean metric if and only if this form Q is positive definite. A simple proof can be found in the first few pages of [Bha07], and, again, the above result can also be found in [Riv03]. This completes the solution to question (1).

To solve question (2), let $x, y \in \sigma$ with barycentric coordinates $(x_i)_{i=0}^n$ and $(y_i)_{i=0}^n$, respectively. Just as before we consider some linear embedding of σ into \mathbb{R}^n and abuse notation by associating x and y with their images in \mathbb{R}^n . Note then that $x = \sum_{i=0}^n x_i v_i$ and $y = \sum_{i=1}^n y_i v_i$. The square of the distance $d_\sigma(x, y)$ between x and y in σ is given by $\langle x - y, x - y \rangle$, where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form defined above. What is left to do is to show how to use equation (2.2) to produce a nice formula to compute $d_\sigma(x, y)$.

Define the quadratic form Q as above and note that, with respect to our basis $\{w_i\}_{i=1}^n$, we can express Q as an $(n \times n)$ symmetric matrix by

$$(2.3) \quad Q_{ij} = (\langle w_i, w_j \rangle)_{ij} = \left(\frac{1}{2} (\gamma_{0i}^2 + \gamma_{0j}^2 - \gamma_{ij}^2) \right)_{ij}.$$

Recall that, by the definition of barycentric coordinates,

$$(2.4) \quad x_0 = 1 - \sum_{i=1}^n x_i \quad \text{and} \quad y_0 = 1 - \sum_{i=1}^n y_i.$$

With the help of equation (2.4) we compute

$$(2.5) \quad \begin{aligned} x - y &= \sum_{i=0}^n (x_i - y_i) v_i = (x_0 - y_0) v_0 + \sum_{i=1}^n (x_i - y_i) v_i \\ &= - \sum_{i=1}^n (x_i - y_i) v_0 + \sum_{i=1}^n (x_i - y_i) v_i = \sum_{i=1}^n (x_i - y_i) w_i. \end{aligned}$$

Combining equations (2.3) and (2.5) yields

$$(2.6) \quad \begin{aligned} \langle x - y, x - y \rangle &= \left\langle \sum_{i=1}^n (x_i - y_i)w_i, \sum_{j=1}^n (x_j - y_j)w_j \right\rangle \\ &= \sum_{i,j=1}^n (x_i - y_i)(x_j - y_j)\langle w_i, w_j \rangle = [x - y] \cdot Q[x - y] \end{aligned}$$

where “ \cdot ” represents the standard Euclidean inner product, and where $[x - y]$ is the vector in \mathbb{R}^n defined by $[x - y] = (x_i - y_i)_{i=1}^n$. Both the matrix Q and the vector $[x - y]$ are expressed using only the barycentric coordinates of x and y and the edge lengths assigned to σ . So, using equation (2.6), computing distances in σ requires only matrix multiplication.

3. THE MINIMAL ALLOWABLE EDGE LENGTH WHEN ALL OTHER EDGES HAVE LENGTH 1

Let σ be as above, and assume that all edges of σ have length 1 except one edge whose length we will denote by α . By symmetry, let e_{0n} be the edge with length α , i.e. $\gamma_{0n} = \alpha$. The question is, for what values of α does σ admit an affine isometric embedding into \mathbb{R}^n ? When $n = 2$ it is easy to see that $0 < \alpha < 2$, and for $n = 3$ one observes that $0 < \alpha < \sqrt{3}$. But it starts to get a little more subtle² once $n \geq 4$. Note that the quadratic form Q from Section 2 is

$$Q(\alpha) = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\alpha^2 \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} & \frac{1}{2}\alpha^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 & \frac{1}{2}\alpha^2 \\ \frac{1}{2}\alpha^2 & \frac{1}{2}\alpha^2 & \cdots & \frac{1}{2}\alpha^2 & \alpha^2 \end{bmatrix}$$

We need to find the values of α for which $Q(\alpha)$ is positive definite. We first need a Lemma:

Lemma 1. *Let A_n and B_n denote the $n \times n$ matrices*

$$A_n = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{bmatrix}, \quad B_n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \cdots & \vdots \\ \frac{1}{2} & \cdots & 1 & \frac{1}{2} \end{bmatrix}$$

Then $\det(A_n) = \frac{n+1}{2^n}$ and $\det(B_n) = \frac{(-1)^{n+1}}{2^n}$.

Proof. The proof proceeds by (simultaneous) induction on n . The base cases are easily checked and left to the reader.

Let us first compute $\det(A_n)$. Add negative of the first row to the n^{th} row, which does not change the determinant. Then cofactor expansion along the new n^{th} row, along with using both

²When doing some research for [Min13], I once assumed $\alpha = \frac{3}{2}$ would always work. To my surprise I found out that this leads to a degenerate simplex when $n = 9$, and does not lead to a realizable Euclidean simplex for all larger dimensions.

portions of the induction hypothesis, yields

$$\begin{aligned}\det(A_n) &= (-1)^{n+1} \left(\frac{-1}{2}\right) \det(B_{n-1}) + \frac{1}{2} \det(A_{n-1}) \\ &= (-1)^{n+1} \left(\frac{-1}{2}\right) \frac{(-1)^n}{2^{n-1}} + \left(\frac{1}{2} \cdot \frac{n}{2^{n-1}}\right) \\ &= \frac{1}{2^n} + \frac{n}{2^n} = \frac{n+1}{2^n}.\end{aligned}$$

To compute $\det(B_n)$, add negative of row 1 to row n . The only term in the new n^{th} row which is not 0 is the second to last term, and it is $\frac{1}{2}$. Then cofactor expansion along the last row gives

$$\det(B_n) = -\frac{1}{2} \det(B_{n-1}) = -\frac{1}{2} \frac{(-1)^n}{2^{n-1}} = \frac{(-1)^{n+1}}{2^n}.$$

□

With the aid of Lemma 1 we are now prepared to prove the following Theorem:

Theorem 2. *The quadratic form $Q(\alpha)$ is positive definite if and only if $0 < \alpha < \sqrt{\frac{2n}{n-1}}$.*

Remark 3. Note that $\sqrt{\frac{2n}{n-1}}$ is a decreasing function, and $\lim_{n \rightarrow \infty} \sqrt{\frac{2n}{n-1}} = \sqrt{2}$. So all values of α with $0 < \alpha < \sqrt{2}$ always lead to a Euclidean simplex. This fact is used in [Min13].

Proof of Theorem 2. By Lemma 1 all of the minors of $Q(\alpha)$ which contain the $(1, 1)$ entry are positive. Thus, $Q(\alpha)$ is positive definite if and only if $\det(Q) > 0$. To compute $\det(Q)$, factor an α^2 out of both the n^{th} column and the n^{th} row. This yields

$$\det(Q) = \alpha^4 \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{\alpha^2} \end{vmatrix}$$

Note that we are assuming that $\alpha > 0$ since it is the side length of a non-degenerate simplex.

As in the proof of Lemma 1 we add negative the first row to the n^{th} row and then use cofactor expansion along the n^{th} row to obtain

$$\begin{aligned}\det(Q) &= \alpha^4 \left((-1)^{n+1} \left(-\frac{1}{2}\right) \det(B_{n-1}) + \left(\frac{1}{\alpha^2} - \frac{1}{2}\right) \det(A_{n-1}) \right) \\ &= \alpha^4 \left((-1)^n \left(\frac{1}{2}\right) \frac{(-1)^n}{2^{n-1}} + \left(\frac{1}{\alpha^2} - \frac{1}{2}\right) \frac{n}{2^{n-1}} \right) \\ (3.1) \quad &= \alpha^2 \left(\frac{\alpha^2(1-n)}{2^n} + \frac{n}{2^{n-1}} \right)\end{aligned}$$

where the notation A_n and B_n comes from Lemma 1. We then see from equation (3.1) that

$$(3.2) \quad \det(Q) > 0 \quad \iff \quad \frac{\alpha^2(1-n)}{2^n} + \frac{n}{2^{n-1}} > 0 \quad \iff \quad \alpha < \sqrt{\frac{2n}{n-1}}.$$

□

4. VOLUME OF AN n -SIMPLEX VIA EDGE LENGTHS

Theorem 4. Let $\sigma = \langle v_0, v_1, \dots, v_n \rangle$ be an n -dimensional Euclidean simplex with edge lengths $\{\gamma_{ij}\}_{i,j=0}^n$. Let Q be the $n \times n$ matrix defined by

$$Q_{ij} = \left(\frac{1}{2} (\gamma_{0i}^2 + \gamma_{0j}^2 - \gamma_{ij}^2) \right).$$

Then

$$(4.1) \quad \text{Vol}(\sigma) = \frac{1}{n!} \sqrt{\det(Q)}$$

Proof. Let σ , $\langle v_i \rangle_{i=0}^n$, $\{\gamma_{ij}\}_{i,j=1}^n$, and Q be as above. Isometrically embed σ into \mathbb{E}^n , and let $w_i := v_i - v_0$ for all i . Let $W = [w_1 \ w_2 \ \dots \ w_n]$ be the $n \times n$ matrix whose columns are the vectors in $\{w_i\}_{i=1}^n$. It is well-known that $\det(W)$ is the volume of the parallelepiped spanned by the vectors in $\{w_i\}_{i=1}^n$. Thus $\text{Vol}(\sigma) = \frac{1}{n!} \det(W)$. But notice that $W^T W = (w_i \cdot w_j)_{ij} = Q$. So $\det(Q) = \det(W)^2$, which proves the Theorem. \square

Remark 5. Combining Theorem 4 with equation (2.3) produces a very nice formula for *intrinsically* computing the volume of an n -simplex, meaning that the formula depends only on the assigned edge lengths and not on the coordinates of any of the vertices. The current technique for finding such volumes is by using the *Cayley-Menger determinant*. This will be discussed in Section 6. For now, it is worth pointing out that computing the same volume using a Cayley-Menger determinant involves computing a $(n+2) \times (n+2)$ determinant and, in the author's opinion, is much less natural.

In [KLM15] we are interested in knowing the edge length of an equilateral n -simplex whose volume is 1. To compute this, let e_n denote the common edge length of σ . Then $Q = (e_n)^2 A_n$ where A_n is the notation used in Lemma 1. Hence

$$1 = \text{Vol}(\sigma) = \frac{1}{n!} \sqrt{\det(Q)} = \frac{(e_n)^n}{n!} \sqrt{\frac{n+1}{2^n}}.$$

Solving for e_n yields

$$(4.2) \quad e_n = \left(n! \sqrt{\frac{2^n}{n+1}} \right)^{\frac{1}{n}}$$

As a last note, it is interesting to consider $\lim_{n \rightarrow \infty} e_n$. It is not hard to check that

$$\lim_{n \rightarrow \infty} (n!)^{\frac{1}{n}} = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \left(\sqrt{\frac{2^n}{n+1}} \right)^{\frac{1}{n}} = \sqrt{2}$$

and thus $\lim_{n \rightarrow \infty} e_n = \infty$. So, for an equilateral n -simplex to have volume 1, as n approaches infinity the edge lengths must approach infinity as well. The geometric intuition here is to notice that the equilateral n -simplex with unit edge lengths comprises of a smaller percentage of the unit hypercube as n gets larger. Thus, the volume of the simplex decreases in n , and so the edge lengths must increase to make the volume 1.

5. DISTANCE FROM THE BARYCENTER TO THE BOUNDARY OF AN EQUILATERAL SIMPLEX

As a final example of the utility of our matrix Q , let us compute the distance from the barycenter to the boundary of an equilateral simplex. Let σ be an equilateral simplex, so that all edge lengths are the same length $\gamma := \gamma_{ij}$. Let b denote the barycenter of σ , meaning that b has barycentric coordinates $\left(\frac{1}{n+1}\right)_{i=1}^n$. Since σ is equilateral, the distance from b to the boundary of σ is equal to the distance from b to the barycenter of any of the codimension one faces of σ . For convenience, we compute the distance from b to b' , where b' is the barycenter of the codimension one face opposite the vertex v_0 . So b' has barycentric coordinates $(0, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$.

Observe that

$$[b' - b] = \left(\frac{1}{n(n+1)}\right)_{i=1}^n \quad \text{and} \quad Q_{ij} = \begin{cases} \gamma^2 & \text{if } i = j \\ \frac{1}{2}\gamma^2 & \text{if } i \neq j \end{cases}$$

A simple calculation then shows that

$$[b' - b] \cdot Q[b' - b] = \frac{\gamma^2}{n^2(n+1)^2} \left(\frac{n(n+1)}{2}\right) = \frac{\gamma^2}{2n(n+1)}.$$

Thus,

$$(5.1) \quad d_\sigma(b, \partial\sigma) = \frac{\gamma}{\sqrt{2n(n+1)}}$$

Of particular interest is knowing $d_\sigma(b, \partial\sigma)$ when σ is an equilateral simplex with volume 1. Using the notation from Section 4 we have that $\gamma = e_n$, and combining equations (4.2) and (5.1) yields:

$$(5.2) \quad d_\sigma(b, \partial\sigma) = \frac{e_n}{\sqrt{2n(n+1)}} = \left(n! \sqrt{\frac{2^n}{n+1}}\right)^{\frac{1}{n}} \cdot \left(\frac{1}{\sqrt{2n(n+1)}}\right) = \left(\frac{n!}{\sqrt{n^n(n+1)^{n+1}}}\right)^{\frac{1}{n}}$$

6. CAYLEY-MENGER DETERMINANTS AND GROMOV'S \mathcal{K} -CURVATURE QUESTION

Given an oriented n -simplex $\sigma = \langle v_0, v_1, \dots, v_n \rangle$ with associated edge lengths $\{\gamma_{ij}\}$, one could organize this data into an $(n+1) \times (n+1)$ matrix $B = (b_{ij})$ defined by

$$(6.1) \quad b_{ij} = \gamma_{ij}^2$$

where $\gamma_{ii} := 0$.

The first thing to point out is that the matrix Q defined by equation (2.3) and the matrix B in equation (6.1) are "equivalent" in the sense that they are uniquely determined by the exact same data. Moreover, this data (the edge lengths) can be easily recovered from either matrix. Therefore, given Q it is easy to construct B , and vice versa.

The matrix B mentioned above was first considered by Cayley in [Cay41], and independently studied 80 years later by Menger in [Men28]. They used the matrix B to intrinsically compute the volume of σ just as in Theorem 4. This volume formula is:

$$(6.2) \quad \text{Vol}(\sigma) = \left(\frac{(-1)^{n-1}}{2^n(n!)^2} \det(\bar{B})\right)^{\frac{1}{2}}$$

where \bar{B} is the $(n+2) \times (n+2)$ matrix obtained by placing B in the bottom right hand corner, adding a top row of $(0, 1, 1, \dots, 1)$, and a left column of $(0, 1, 1, \dots, 1)^T$. The determinant $\det(\bar{B})$ is often referred to as the *Cayley-Menger determinant*.

As the matrices Q and B are defined using the exact same data, the formulas (4.1) and (6.2) have some similarities. But formula (4.1) is certainly computationally simpler because it requires computing an $n \times n$ determinant instead of an $(n+2) \times (n+2)$ determinant. Of course, one makes a slight sacrifice in the simplicity of the matrix representation when considering Q over B . But in return one gains a simpler volume formula as well as a natural means of computing distances and other geometric quantities in an intrinsic manner.

In closing, we relate the matrix Q with Gromov's \mathcal{K} -curvature problem found in [Gro01]. The following is taken almost directly from [Gro01].

Let (\mathcal{X}, d) be an arbitrary metric space, and let M_r denote the space of positive symmetric $r \times r$ matrices. Let $K_r(\mathcal{X}) \subset M_r$ denote the subset realizable by the distances among r -tuples of points as given by the matrix B in equation (6.1). i.e., the $r \times r$ matrix³ $B = (b_{ij}) \in K_r(\mathcal{X})$ if and only if there exists an r -tuple of points $(x_1, \dots, x_r) \in \mathcal{X}^r$ such that $d(x_i, x_j)^2 = b_{ij}$ for all i, j . Then every subset $\mathcal{K} \subset M_r$ defines the *global \mathcal{K} -curvature class*, which consists of the spaces \mathcal{X} with $K_r(\mathcal{X}) \subset \mathcal{K}$, and the *local \mathcal{K} -curvature class*, where each point $x \in \mathcal{X}$ is required to admit a neighborhood U with $K_r(U) \subset \mathcal{K}$. Gromov's curvature problem is then:

Gromov's Curvature Problem: Given $\mathcal{K} \subset M_r$, describe the spaces \mathcal{X} in the \mathcal{K} -curvature class.

This problem was answered in some very specific cases by Gromov in [Gro01]. When $r = 4$, this problem was solved (in the global setting) by Berg and Nikolaev in [BN08] for CAT(0) spaces, and by Lebedeva and Petrunin in [LP10] for spaces whose curvature is bounded below. But all of these solutions deal with the data in the matrix B and not the actual matrix B itself.

In light of the discussion above, we can equivalently replace the matrix B with the matrix Q when discussing Gromov's curvature problem. But the matrix Q seems to more closely capture the geometry of the points involved. So one could ask the same question but look for answers which intrinsically depend on Q instead of inequalities using the specific distances. For example, one could ask how knowledge of the eigenvalues and associated eigenspaces of such matrices Q effect the geometry of the underlying space \mathcal{X} , and vice versa.

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³Technically, Gromov considers $b_{ij} = d(x_i, x_j)$ instead of $b_{ij} = d(x_i, x_j)^2$. But these two formulations are clearly equivalent.

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