WARPED PRODUCT METRICS ON (COMPLEX) HYPERBOLIC MANIFOLDS

BARRY MINEMYER

Abstract. In this paper we study manifolds of the form $X \setminus Y$, where X denotes either $\mathbb{H}^n$ or $\mathbb{CH}^n$, and Y is a totally geodesic submanifold with arbitrary codimension. The main results that we prove are curvature formulas for warped product metrics on $X \setminus Y$ expressed in spherical coordinates about Y. We also discuss future applications of these formulas.

1. Introduction

1.1. **Main results.** Let $\mathbb{H}^n$ denote (real) n-dimensional hyperbolic space, and $\mathbb{CH}^n$ denote (complex) n-dimensional complex hyperbolic space. In this paper, X will denote either $\mathbb{H}^n$ or $\mathbb{CH}^n$, and Y will denote a totally geodesic submanifold of X. So if $X = \mathbb{H}^n$ then $Y = \mathbb{H}^k$, and if $X = \mathbb{CH}^n$ then Y is either $\mathbb{H}^k$ or $\mathbb{CH}^k$ for some $1 \leq k \leq n$.

Let M be a Riemannian manifold, and N a totally geodesic submanifold of M. We say that the pair (M, N) is *modeled on* (X, Y) if there exist lattices $\Gamma \subset \mathrm{Isom}(X)$ and $\Lambda \subset \mathrm{Isom}(Y)$ such that $M = X/\Gamma$, $N = Y/\Lambda$, and $\Lambda < \Gamma$. We also allow for the possibility that N is disconnected. That is, we allow for multiple lattices $\Lambda < \Gamma$ which correspond to different (disjoint) copies of $\mathbb{H}^k \subset \mathbb{H}^n$ or $\mathbb{H}^k, \mathbb{CH}^k \subset \mathbb{CH}^n$. The author, along with several collaborators, has been undergoing a systematic study of the geometry and topology of manifolds "affiliated" with the pair (M, N). The purpose of this paper is to continue the development of the curvature formulas used in this research.

More specifically, in this paper we develop curvature formulas for warped product metrics on $X \setminus Y$ when the pair (X, Y) is one of $(\mathbb{H}^n, \mathbb{H}^k)$, $(\mathbb{CH}^n, \mathbb{H}^n)$, or $(\mathbb{CH}^n, \mathbb{CH}^k)$. These cases are detailed in Sections 2, 3, and 4, respectively. In each case we write the metric on X in spherical coordinates about Y (Theorems 2.1, 3.1, and 4.1), we consider the corresponding warped product metric where we allow for variable coefficients in the metric tensor (equations (2.2), (3.3), and (4.2)), and we compute formulas for the components of the $(4, 0)$ curvature tensor with respect to these coefficient functions (Theorems 2.2, 3.4, and 4.3). These last three Theorems should be considered the main results of this paper.

1.2. **Applications for these curvature formulas.** While the author believes that the curvature formulas in Theorems 2.2, 3.4, and 4.3 are of their own independent interest, the primary motivation for the development of these curvature formulas is for the following application.

Date: July 17, 2018.

2010 *Mathematics Subject Classification.* Primary 53C20, 53C35; Secondary 53C56, 57R25.

Key words and phrases. complex hyperbolic space, hyperbolic space, totally geodesic submanifold, warped product metric, sectional curvature.

In [GT87] Gromov and Thurston famously construct pinched negatively curved manifolds which do not admit hyperbolic metrics. In this construction they consider pairs (M, N) modeled on $(\mathbb{H}^n, \mathbb{H}^{n-2})$ which satisfy a few special topological and geometric conditions. The pinched negatively curved manifold B which does not admit a hyperbolic metric is then the k-fold cyclic branched cover of M about N (where $k \in \mathbb{N}$ can take all but possibly finitely many values). The difficulty in all of this is showing that B exists, constructing a pinched negatively curved metric on B, and proving that B does not admit a hyperbolic metric.

The author, along with several collaborators, has been investigating whether or not this construction could be extended to the locally symmetric pairs $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$ and $(\mathbb{CH}^2, \mathbb{H}^2)$. In a sequence of forthcoming papers (see [LMMT] and [Min18b]) we will show that in both of these cases the k-fold cyclic ramified cover of M about N *does* admit a negatively curved Riemannian metric whose sectional curvature is both bounded from below and bounded away from zero (assuming that this cyclic ramified covering exists as a smooth manifold). The constructions of these Riemannian metrics are dependent on the curvature formulas proved in Theorems 3.4 and 4.3 below.

A second, more minor, application of these curvature formulas is the following. In [AP16] Avramidi and Phan prove that if M is a complete finite volume Riemannian manifold with bounded nonpositive sectional curvature, then the "thin part" of M (the portion of M "heading off" toward the cusp) can only have nonzero homology up to dimension $\lfloor \frac{n}{2} \rfloor - 1$. In particular, if $n = 5$, then the thin part of M must be aspherical. In [Min18a] the author uses the formulas for the case $(\mathbb{CH}^3, \mathbb{CH}^1)$ to show that this result does not, in some sense, generalize to distributions. Given any manifold of the form $M \setminus N$, where (M, N) is modeled on $(\mathbb{CH}^3, \mathbb{CH}^1)$, there exists a Riemannian metric g and a non-integrable 5-dimensional distribution $\mathcal{D}$ where g restricted to $\mathcal{D}$ satisfies all of the conditions above. The ends of $M \setminus N$ are of the form $\mathbb{S}^3 \times \mathbb{R}^2$, and so if $\mathcal{D}$ were integrable then the corresponding submanifold with this inherited metric would violate the results in [AP16]. Of course, this metric can more generally be constructed on mainfolds $M \setminus N$ modeled on $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$.

One last remark about these curvature formulas. In [Bel12], [Bel11], and [Min17] it is proved that the manifold $M \setminus N$, where (M, N) is modeled on one of $(\mathbb{H}^n, \mathbb{H}^{n-2})$, $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$, or $(\mathbb{CH}^2, \mathbb{H}^2)$, admits a complete, finite volume, negatively curved Riemannian metric. The curvature formulas developed in this paper generalize the curvature formulas computed and used in these three articles.

1.3. **Obstructions to $M \setminus N$ admitting a complete, finite volume Riemannian metric of negative sectional curvature.** Consider the finite volume manifold $M \setminus N$. The three cases where N has codimension two in M are modeled on one of $(\mathbb{H}^n, \mathbb{H}^{n-2})$, $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$, or $(\mathbb{CH}^2, \mathbb{H}^2)$. In all of these cases, the manifold $M \setminus N$ admits a complete, finite volume Riemannian metric whose sectional curvature is bounded above by a negative constant ([Bel12], [Bel11], and [Min17]).

In general, when the codimension of N is greater than two, the manifold $M \setminus N$ should not admit a complete, finite volume, negatively curved metric because it is not aspherical. This fact should be realized in the curvature equations in Theorems 2.2, 3.4, and 4.3. More specifically, there should be an equation(s) which obstructs such a metric, but this (these) curvature equations should vanish when N has codimension two.

In all cases except one "exceptional case" the obstruction is a sectional curvature equation of the form

$$\frac{1}{v^2} - \left(\frac{v'}{v}\right)^2 \tag{1.1}$$

where $v : \mathbb{R} \to \mathbb{R}$ is a positive, increasing real-valued function. One easily checks that equation (1.1) is nonpositive if and only if $1 \leq (v')^2$. But for the Riemannian metric to have any chance of having finite volume one needs $\lim_{r\to-\infty} v'(r) = 0$.

The one exceptional case is when (M, N) is modeled on $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$. Here, all curvature equations of the form (1.1) vanish, and so this obstruction is considerably more subtle. It should be noted that the vanishing of (1.1) is what leads to the metric in the second application mentioned above. A detailed discussion about this situation is given in Subsection 4.6.

1.4. **Layout of this paper.** In Section 2 we study manifolds of the form $\mathbb{H}^n \setminus \mathbb{H}^k$, in Section 3 we consider $\mathbb{CH}^n \setminus \mathbb{H}^n$, and in Section 4 we analyze $\mathbb{CH}^n \setminus \mathbb{CH}^k$. In Section 3 we restrict our attention to $\mathbb{CH}^3 \setminus \mathbb{H}^3$ and in Section 4 we restrict to $\mathbb{CH}^5 \setminus \mathbb{CH}^2$ for ease of exposition. In each case, these are the smallest choices for n and k which capture all of the different formulas for the curvature tensor (with respect to the frames chosen in each Section). That is, from these cases one knows all of curvature formulas for general $\mathbb{CH}^n \setminus \mathbb{H}^n$ and $\mathbb{CH}^n \setminus \mathbb{CH}^k$. Also, notice that we only consider $\mathbb{CH}^n \setminus \mathbb{H}^n$ in Section 3 instead of the more general $\mathbb{CH}^n \setminus \mathbb{H}^k$. The reason for this is due to simplicity: in general there are several ways that $\mathbb{H}^k$ can sit inside of $\mathbb{CH}^n$ which requires a case-by-case analysis. But in all situations this copy of $\mathbb{H}^k$ is contained in a copy of $\mathbb{H}^n$, and then one can apply our formulas here to $\mathbb{CH}^n \setminus \mathbb{H}^n$. Section 5 is a short Section on some known formulas that are referenced throughout the paper, and Section 6 is devoted to computing values for Lie brackets from Section 3.

We end this Section with the following two remarks which deal with notational differences between this paper and references [Bel12], [Bel11], and [Min17].

Remark 1.1. In this paper we scale the complex hyperbolic metric to have sectional curvatures in the interval $[-4, -1]$, whereas in the previous three references the curvatures were scaled to $\left[-1, -\frac{1}{4}\right]$. To adjust the formulas in [Bel12], [Bel11], and [Min17], one simply multiplies the warping functions h, v, and h_r by $\frac{1}{2}$. With this adjustment (and the following Remark), one sees that the formulas in these references agree with the codimension two versions of the formulas in Theorems 2.2, 3.4, and 4.3.

Remark 1.2. Another major notational difference between this paper and the papers [Bel11] and [Bel12] is the formula used for the curvature tensor. Let g be a Riemannian metric with Levi-Civita connection ∇, and let W, X, Y, and Z be vector fields. In this paper we follow [doC92] and use the notation

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \tag{1.2}$$

for the curvature tensor R of g. The negative of this formula is used in [Bel11] and [Bel12]. So, in particular, the $(4,0)$-curvature tensor $\langle R(X,Y)Z, W\rangle_g$ in this paper is equivalent to $\langle R(X,Y)W, Z\rangle_g$ in [Bel11] and [Bel12].

2. Curvature formulas for warped product metrics on $\mathbb{H}^n \setminus \mathbb{H}^k$

2.1. **Expressing the metric in $\mathbb{H}^n$ in spherical coordinates about $\mathbb{H}^k$.** Let us first note that in Subsections 2.1, 3.1, and 4.1 we closely follow the notation and terminology used in [Bel11].

Let $\mathbf{h_n}$ denote the hyperbolic metric on $\mathbb{H}^n$. Since $\mathbb{H}^k$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{H}^n$, there exists an orthogonal projection map $\pi : \mathbb{H}^n \to \mathbb{H}^k$. This map π is a fiber bundle whose fibers are totally geodesic $(n-k)$-planes.

For $r > 0$ let $E(r)$ denote the r-neighborhood of $\mathbb{H}^k$. Then $E(r)$ is a real hypersurface in $\mathbb{H}^n$, and consequently we can decompose $\mathbf{h_n}$ as

$$\mathbf{h_n} = (\mathbf{h_n})_r + dr^2$$

where $(\mathbf{h_n})_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r : E(r) \to \mathbb{H}^k$ denote the restriction of π to $E(r)$. Note that π_r is an $\mathbb{S}^{n-k-1}$-bundle whose fiber over any point $q \in \mathbb{H}^k$ is the $(n-k-1)$-sphere of radius r in the totally geodesic $(n-k)$-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$.

It is well known (see [Bel12] or [GT87] when $k = n-2$ and [Ont15] for general k) that for an appropriate identification of $E(r) \cong \mathbb{H}^k \times \mathbb{S}^{n-k-1}$ the metric $(\mathbf{h_n})_r$ can be written as

$$(\mathbf{h_n})_r = \cosh^2(r)\mathbf{h_k} + \sinh^2(r)\sigma_{\mathbf{n-k-1}}$$

where $\mathbf{h_k}$ denotes the hyperbolic metric on $\mathbb{H}^k$ and $\sigma_{\mathbf{n-k-1}}$ denotes the round metric on the unit sphere $\mathbb{S}^{n-k-1}$. Note that $(\mathbf{h_n})_r$ restricted to $\mathcal{H}(r)$ is $\cosh^2(r)\mathbf{h_k}$ and $(\mathbf{h_n})_r$ restricted to $\mathcal{V}(r)$ is $\sinh^2(r)\sigma_{\mathbf{n-k-1}}$. We summarize this in the following Theorem.

Theorem 2.1. *The hyperbolic manifold $\mathbb{H}^n \setminus \mathbb{H}^k$ can be written as $E \times (0,\infty)$ where $E \cong \mathbb{H}^k \times \mathbb{S}^{n-k-1}$ equipped with the metric*

$$\mathbf{h_n} = \cosh^2(r)\mathbf{h_k} + \sinh^2(r)\sigma^{n-k-1} + dr^2. \tag{2.1}$$

2.2. **The warped product metric and curvature formulas.** For some positive, increasing real-valued functions $h, v : (0,\infty) \to \mathbb{R}$ define

$$\lambda_{\mathbf{r}} := h^2(r)\mathbf{h_k} + v^2(r)\sigma_{\mathbf{n-k-1}} \qquad \text{and} \qquad \lambda := \lambda_{\mathbf{r}} + dr^2. \tag{2.2}$$

Of course, $\lambda = \mathbf{h_n}$ when $h = \cosh(r)$ and $v = \sinh(r)$.

Fix $p \in E(r)$ for some r and let $q = \pi(p) \in \mathbb{H}^k$. Let $\{\check{X}_i\}_{i=1}^k$ be an orthonormal frame of $\mathbb{H}^k$ near q which satisfies $[\check{X}_i, \check{X}_j]_q = 0$ for all $1 \leq i,j \leq k$. These vector fields can be extended to a collection of orthogonal vector fields $\{X_i\}_{i=1}^k$ in a neighborhood of p via the inclusion $\mathbb{H}^k \to E \times (0,\infty)$. Analogously, define an orthonormal frame $\{\check{X}_j\}_{j=k+1}^{n-1}$ of $\mathbb{S}^{n-k-1}$ near (the projection of) p which satisfies $[\check{X}_i, \check{X}_j]_p = 0$ for all $k+1 \leq i,j \leq n-1$, and extend this frame to vector fields $\{X_j\}_{j=k+1}^{n-1}$ in a neighborhood of p via the inclusion $\mathbb{S}^{n-k-1} \to E \times (0,\infty)$. Lastly, let $X_n = \frac{\partial}{\partial r}$.

The orthogonal collection of vector fields $\{X_i\}_{i=1}^n$ satisfies the following:

(1) $\langle X_i, X_i \rangle_\lambda = h^2$ for $1 \leq i \leq k$.
(2) $\langle X_i, X_i \rangle_\lambda = v^2$ for $k+1 \leq i \leq n-1$.

(3) $\langle X_n, X_n\rangle_\lambda = 1$.
(4) $[X_i, X_j]_p = 0$ for all i, j.

It should be noted that property (4) is special to the real hyperbolic case and will not be true in Sections 3 and 4 below.

Now define the corresponding orthonormal frame near p by $Y_i = \frac{1}{h}X_i$ for $1 \leq i \leq k$, $Y_j = \frac{1}{v}X_j$ for $k+1 \leq j \leq n-1$, and $Y_n = X_n$. This frame satisfies the property that $[Y_i, Y_j]_p = 0$ for $1 \leq i, j \leq n-1$. We can then apply formulas (5.4) through (5.7) to write the (4,0) curvature tensor R_λ in terms of R_{λ_r} as follows, where $1 \leq a, b \leq k$ and $k+1 \leq c, d \leq n-1$.

$$K_\lambda(Y_a, Y_b) = K_{\lambda_r}(Y_a, Y_b) - \left(\frac{h'}{h}\right)^2 \qquad K_\lambda(Y_c, Y_d) = K_{\lambda_r}(Y_c, Y_d) - \left(\frac{v'}{v}\right)^2$$
$$K_\lambda(Y_a, Y_c) = K_{\lambda_r}(Y_a, Y_c) - \frac{h'v'}{hv} \qquad K_\lambda(Y_a, Y_n) = -\frac{h''}{h} \qquad K_\lambda(Y_c, Y_n) = -\frac{v''}{v}.$$

In the above equations, we use the notation

$$K(X, Y) = \langle R(X, Y)X, Y\rangle$$

to denote the sectional curvature of the 2-plane spanned by X and Y. The above equations are the only terms that appear (up to the symmetries of the curvature tensor). So, in particular, all mixed terms of R_λ are identically zero.

Now, the $(4,0)$ curvature tensor R_{λ_r} is simple to calculate. Since both $h(r)\mathbb{H}^k$ and $v(r)\mathbb{S}^{n-k-1}$ have constant curvature, and $h(r)\mathbb{H}^k \times v(r)\mathbb{S}^{n-k-1}$ is metrically a product, we have that for $1 \leq a, b, \leq k$ and $k+1 \leq c, d \leq n-1$:

$$K_{\lambda_r}(Y_a, Y_b) = -\frac{1}{h^2} \qquad K_{\lambda_r}(Y_c, Y_d) = \frac{1}{v^2} \qquad K_{\lambda_r}(Y_a, Y_c) = 0.$$

Putting this all together yields the following.

Theorem 2.2. *Up to the symmetries of the curvature tensor, the only nonzero terms of the* $(4,0)$ *curvature tensor* R_λ *are:*

$$K_\lambda(Y_a, Y_b) = -\frac{1}{h^2} - \left(\frac{h'}{h}\right)^2 \qquad K_\lambda(Y_c, Y_d) = \frac{1}{v^2} - \left(\frac{v'}{v}\right)^2$$
$$K_\lambda(Y_a, Y_c) = -\frac{h'v'}{hv} \qquad K_\lambda(Y_a, Y_n) = -\frac{h''}{h} \qquad K_\lambda(Y_c, Y_n) = -\frac{v''}{v}$$

where $1 \leq a, b, \leq k$ *and* $k+1 \leq c, d \leq n-1$.

One easily checks that plugging in the values $v(r) = \sinh(r)$ and $h(r) = \cosh(r)$ gives all sectional curvatures of -1.

3. Curvature formulas for warped product metrics on $\mathbb{CH}^n \setminus \mathbb{H}^n$

As mentioned in the Introduction, for simplicity we are going to restrict ourselves to the case when $n = 3$. This is exactly the smallest dimension which captures every nonzero component of the curvature tensor, and so nothing is lost with this restriction (see the comments after Theorem 3.4 for more discussion).

3.1. **Expressing the metric in $\mathbb{CH}^3$ in spherical coordinates about $\mathbb{H}^3$.** Let $\mathbf{c_3}$ denote the complex hyperbolic metric on $\mathbb{CH}^3$ normalized to have constant holomorphic sectional curvature -4. Since $\mathbb{H}^3$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{CH}^3$, there exists an orthogonal projection map $\pi : \mathbb{CH}^3 \to \mathbb{H}^3$. This map π is a fiber bundle whose fibers are totally real totally geodesic 3-planes, and therefore have constant sectional curvature -1.

For $r > 0$ let $E(r)$ denote the r-neighborhood of $\mathbb{H}^3$. Then $E(r)$ is a real hypersurface in $\mathbb{CH}^3$, and consequently we can decompose $\mathbf{c_3}$ as

$$\mathbf{c_3} = (\mathbf{c_3})_r + dr^2$$

where $(\mathbf{c_3})_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r : E(r) \to \mathbb{H}^3$ denote the restriction of π to $E(r)$. Note that π_r is an $\mathbb{S}^2$-bundle whose fiber over any point $q \in \mathbb{H}^3$ is the 2-sphere of radius r in the totally real totally geodesic 3-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the 2-sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$.

For $r, s > 0$ there exists a diffeomorphism $\phi_{sr} : E(s) \to E(r)$ induced by the geodesic flow along the totally real totally geodesic 3-planes orthogonal to $\mathbb{H}^3$. Fix $p \in E(r)$ arbitrary, let $q = \pi(p) \in \mathbb{H}^3$, and let γ be the unit speed geodesic such that $\gamma(0) = q$ and $\gamma(r) = p$. In what follows, all computations are considered in the tangent space $T_pE(r)$.

Note that $\mathcal{V}(r)$ is tangent to both $E(r)$ and the totally real totally geodesic 3-plane $\pi^{-1}(q)$. Then since $\pi^{-1}(q)$ is preserved by the geodesic flow, we have that $d\phi_{sr}$ takes $\mathcal{V}(s)$ to $\mathcal{V}(r)$. Since $exp_p^{-1}\left(\pi^{-1}(q)\right)$ is a totally real 3-plane, there exists a suitable identification $\pi^{-1}(q) \cong \mathbb{S}^2 \times (0, \infty)$ where the metric $\mathbf{c_3}$ restricted to $\pi^{-1}(q)$ can be written as

$$\sinh^2(r)\sigma^2 + dr^2.$$

Here, σ^2 is the round metric on the unit 2-sphere.

Let

$$\check{X}_4 = \frac{\partial}{\partial\theta} \qquad\qquad \check{X}_5 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} \tag{3.1}$$

be an orthonormal frame on a neighborhood of (the projection of) p in $\mathbb{S}^2$, and extend these to orthogonal vector fields $\{X_4, X_5\}$ on $\pi^{-1}(q)$ via the inclusion $\mathbb{S}^2 \to \pi^{-1}(q)$. Note that both X_4 and X_5 are invariant under $d\phi_{sr}$. Let $X_6 = \frac{\partial}{\partial r}$.

Let J denote the complex structure on $\mathbb{CH}^3$. It is well known that J_p preserves complex subspaces in $T_p\mathbb{CH}^3$ and maps real subspaces into their orthogonal complement. Since (X_4, X_5, X_6) spans a real 3-plane in $T_p\mathbb{CH}^3$, its orthogonal complement $\mathcal{H}_p(r)$ is spanned by (JX_4, JX_5, JX_6). In what follows we define vector fields X_1, X_2, and X_3 which are just scaled copies of JX_4, JX_5, and JX_6, respectively.

3.1.1. *The vector fields X_1 and X_2.* First note that (JX_4, X_6) spans a real 2-plane in $T_p\mathbb{CH}^3$ (since its J-image is contained in its orthogonal complement). So $P = \exp(\text{span}(JX_4, X_6))$ is a totally real totally geodesic 2-plane in $\mathbb{CH}^3$ which intersects $\mathbb{H}^3$ orthogonally. Since this intersection is orthogonal, P is preserved by the geodesic flow ϕ. Therefore, span (JX_4) is preserved by $d\phi$.

The set $P \cap \mathbb{H}^3$ is a (real) geodesic. Let $\alpha(s)$ denote this geodesic parameterized with respect to arc length so that $\alpha(0) = q$. Then define $(X_1)_p = (d\pi)_p^{-1}\alpha'(0)$. There exists a positive real-valued function $a(r, s)$ so that the metric $\mathbf{c_3}$ restricted to P is of the form $dr^2 + a^2(r, s)ds^2$. But since $\mathbb{R}$ acts by isometries on P via translation along α, the function $a(r,s)$ is independent of s. Then since the curvature of a real 2-plane is -1, we have that $a(r) = \cosh(r)$.

We analogously define X_2 by replacing X_4 with X_5 in the above description. All conclusions follow in an identical manner. Thus, we can write the metric $\mathbf{c_3}$ restricted to $exp_p(X_1, X_2, X_6)$ as

$$\cosh^2(r)(dX_1^2 + dX_2^2) + dr^2.$$

3.1.2. *The vector field* X_3. This is also mostly analogous to the definition of X_1. But this time note that (JX_6, X_6) spans a complex line in $T_p\mathbb{CH}^3$ (since it is preserved by its J-image). So $Q = \exp_p(\mathrm{span}\,(JX_6, X_6))$ is a complex geodesic in $\mathbb{CH}^3$ which intersects $\mathbb{H}^3$ orthogonally. Since this intersection is orthogonal, Q is preserved by the geodesic flow ϕ. Therefore, $\mathrm{span}\,(JX_6)$ is preserved by $d\phi$.

The set $Q \cap \mathbb{H}^3$ is a (real) geodesic. Let $\beta(t)$ denote this geodesic parameterized with respect to arc length so that $\beta(0) = q$. Then define $(X_3)_p = (d\pi)^{-1}\beta'(0)$. There exists a positive real-valued function $b(r,t)$ so that the metric $\mathbf{c_3}$ restricted to Q is of the form $dr^2 + b^2(r,t)dt^2$. But since $\mathbb{R}$ acts by isometries on Q via translation along β, the function $b(r,t)$ is independent of t. Then since the curvature of a complex geodesic is -4, we have that $b(r) = \cosh(2r)$.

3.1.3. *Conclusion.*

Theorem 3.1. *The complex hyperbolic manifold $\mathbb{CH}^3 \setminus \mathbb{H}^3$ can be written as $E \times (0,\infty)$ where $E \cong \mathbb{H}^3 \times \mathbb{S}^2$ equipped with the metric*

$$\mathbf{c_3} = \cosh^2(r)(dX_1^2 + dX_2^2) + \cosh^2(2r)dX_3^2 + \sinh^2(r)(dX_4^2 + dX_5^2) + dr^2. \tag{3.2}$$

In equation (3.2), dX_1 through dX_5 denote the covector fields dual to the vector fields X_1 through X_5, respectively. Lastly, notice that $dX_1^2 + dX_2^2$ is the hyperbolic metric with constant sectional curvature -1, and $dX_4^2 + dX_5^2$ is the spherical metric with constant sectional curvature 1.

3.2. **The warped product metric and curvature formulas in $\mathbb{CH}^3 \setminus \mathbb{H}^3$.** For some positive, increasing real-valued functions $h, h_r, v : (0,\infty) \to \mathbb{R}$ define

$$\mu_{\mathbf{r}} := h^2(r)(dX_1^2 + dX_2^2) + h_r^2(r)dX_3^2 + v^2(r)(dX_4^2 + dX_5^2)$$

and

$$\mu := \mu_{\mathbf{r}} + dr^2. \tag{3.3}$$

Of course, $\mu = \mathbf{c_3}$ when $h = \cosh(r)$, $h_r = \cosh(2r)$, and $v = \sinh(r)$.

Define an orthonormal basis $\{Y_i\}_{i=1}^6$ with respect to μ by

$$\begin{aligned} Y_1 &= \frac{1}{h}X_1 & Y_2 &= \frac{1}{h}X_2 & Y_3 &= \frac{1}{h_r}X_3 \\ Y_4 &= \frac{1}{v}X_4 & Y_5 &= \frac{1}{v}X_5 & Y_6 &= X_6. \end{aligned} \tag{3.4}$$

Our goal is to compute formulas for the components of the $(4,0)$ curvature tensor R_μ in terms of the warping functions h, h_r, and v (this is the content of Theorem 3.4). As a first step, we need to compute the components of the $(4,0)$ curvature tensor $R_{\mathbf{c_3}}$ of the complex hyperbolic metric with respect to the orthonormal basis given above. We can do this with the help of formula (5.1). To use this formula note that, by construction, we have that $JY_4 = Y_1$, $JY_5 = Y_2$, and $JY_6 = Y_3$ (again, when the metric is $\mathbf{c_3}$, that is, when $h = \cosh(r)$, $h_r = \cosh(2r)$, and $v = \sinh(r)$). Lastly, we use the notation

$$R^{\mathbf{c_3}}_{ijkl} := \langle R_{\mathbf{c_3}}(Y_i, Y_j)Y_k, Y_l\rangle_{\mathbf{c_3}}.$$

Then, up to the symmetries of the curvature tensor, the nonzero components of the $(4,0)$ curvature tensor $R_{\mathbf{c_3}}$ are

$$-4 = R^{\mathbf{c_3}}_{1414} = R^{\mathbf{c_3}}_{2525} = R^{\mathbf{c_3}}_{3636} \tag{3.5}$$

$$\begin{aligned} -1 &= R^{\mathbf{c_3}}_{1212} = R^{\mathbf{c_3}}_{1313} = R^{\mathbf{c_3}}_{1515} = R^{\mathbf{c_3}}_{1616} = R^{\mathbf{c_3}}_{2323} = R^{\mathbf{c_3}}_{2424} \\ &= R^{\mathbf{c_3}}_{2626} = R^{\mathbf{c_3}}_{3434} = R^{\mathbf{c_3}}_{3535} = R^{\mathbf{c_3}}_{4545} = R^{\mathbf{c_3}}_{4646} = R^{\mathbf{c_3}}_{5656} \end{aligned} \tag{3.6}$$

$$-2 = R^{\mathbf{c_3}}_{1425} = R^{\mathbf{c_3}}_{1436} = R^{\mathbf{c_3}}_{2536} \tag{3.7}$$

$$-1 = R^{\mathbf{c_3}}_{1245} = R^{\mathbf{c_3}}_{1346} = R^{\mathbf{c_3}}_{2356} = R^{\mathbf{c_3}}_{1524} = R^{\mathbf{c_3}}_{1634} = R^{\mathbf{c_3}}_{2635}. \tag{3.8}$$

3.3. **Lie brackets.** We now need to compute the values of the Lie brackets of the orthogonal basis $\{X_i\}_{i=1}^6$. A first observation is that, by construction, each of these vector fields is invariant under the flow of $\frac{\partial}{\partial r}$. This implies that $[X_i, X_6] = 0$ for all $1 \leq i \leq 6$. From this we can deduce that

$$[Y_1, Y_6] = \frac{h'}{h}Y_1 \qquad [Y_2, Y_6] = \frac{h'}{h}Y_2 \qquad [Y_3, Y_6] = \frac{h_r'}{h_r}Y_3$$

$$[Y_4, Y_6] = \frac{v'}{v}Y_4 \qquad [Y_5, Y_6] = \frac{v'}{v}Y_5.$$

Next, we know that each Lie bracket is tangent to the level surfaces of r. Thus, for all $1 \leq i, j \leq 6$, the Lie bracket $[X_i, X_j]$ has no X_6 term. For all $1 \leq i, j, k \leq 5$ define structure constants c^k_{ij} by

$$[X_i, X_j] = \sum_{k=1}^{5} c^k_{ij} X_k. \tag{3.9}$$

Two quick observations about the structure constants. The first is that $c^k_{ij} = -c^k_{ji}$ due to the anti-symmetry of the Lie bracket. The second observation is about the values of c^4_{45} and c^5_{45}. Recall the definitions for $\check{X}_4$ and $\check{X}_5$ from equation (3.1). Then

$$\left[\check{X}_4, \check{X}_5\right] = \left[\frac{\partial}{\partial\theta}, \frac{1}{\sin(\theta)}\frac{\partial}{\partial\psi}\right] = \frac{-\cos(\theta)}{\sin^2(\theta)}\frac{\partial}{\partial\psi} = -\cot(\theta)\check{X}_5. \tag{3.10}$$

We therefore conclude that $c^4_{45} = 0$ and $c^5_{45} = -\cot(\theta)$.

The following Theorem gives almost a full description of the values of the Lie brackets. Some quantities are only defined up to sign, but this is sufficient to compute the curvature formulas in Theorem 3.4. The interested reader can find the proof of Theorem 3.2 in Section 6.

Theorem 3.2. *The values for the Lie brackets in equation* (3.9) *are*

$$[X_1, X_2] = \pm X_1 \qquad [X_1, X_3] = \mp\cot(\theta)X_2 + X_4$$

$$[X_1, X_4] = X_3 \mp X_5 \qquad [X_1, X_5] = -\cot(\theta)X_2 \pm X_4$$

$$[X_2, X_3] = \pm\cot(\theta)X_1 + X_5 \qquad [X_2, X_4] = 0$$

$$[X_2, X_5] = \cot(\theta)X_1 + X_3 \qquad [X_3, X_4] = -X_1 \pm \cot(\theta)X_5$$
$$[X_3, X_5] = -X_2 \mp \cot(\theta)X_4 \qquad [X_4, X_5] = -\cot(\theta)X_5$$

In the above equations, all of the $\pm$ and $\mp$ signs are related. For example, if it is the case that $[X_1, X_2] = X_1$, then $[X_1, X_4] = X_3 - X_5$ and so on.

3.4. **The Levi-Civita connection and formulas for the (4,0) curvature tensor R_μ.** In this Subsection we first compute the Levi-Civita connection ∇ associated to the metric μ with respect to the frame $(Y_i)_{i=1}^6$. The difficult part in all of this is computing the Lie brackets in Theorem 3.2. From there it is now a simple calculation using formula (5.3) to prove the following Theorem.

Theorem 3.3. *The Levi-Civita connection ∇ compatible with μ is determined by the 36 equations*

- $\nabla_{Y_1} Y_1 = \mp \frac{1}{h} Y_2 - \frac{h'}{h} Y_6$
- $\nabla_{Y_3} Y_4 = -\frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} - \frac{v}{hh_r}\right) Y_1 \pm \frac{1}{h_r}\cot(\theta) Y_5$
- $\nabla_{Y_1} Y_2 = \pm \frac{1}{h} Y_1$
- $\nabla_{Y_3} Y_5 = -\frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} - \frac{v}{hh_r}\right) Y_2 \mp \frac{1}{h_r}\cot(\theta) Y_4$
- $\nabla_{Y_1} Y_3 = \frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_4$
- $\nabla_{Y_4} Y_1 = -\frac{1}{2}\left(\frac{h}{h_r v} + \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_3$
- $\nabla_{Y_1} Y_4 = -\frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_3 \mp \frac{1}{h} Y_5$
- $\nabla_{Y_4} Y_2 = 0$
- $\nabla_{Y_1} Y_5 = \pm \frac{1}{h} Y_4$
- $\nabla_{Y_4} Y_3 = \frac{1}{2}\left(\frac{h}{h_r v} + \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_1$
- $\nabla_{Y_2} Y_1 = 0$
- $\nabla_{Y_4} Y_4 = -\frac{v'}{v} Y_6$
- $\nabla_{Y_2} Y_2 = -\frac{h'}{h} Y_6$
- $\nabla_{Y_4} Y_5 = 0$
- $\nabla_{Y_2} Y_3 = \frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_5$
- $\nabla_{Y_5} Y_1 = \frac{1}{v}\cot(\theta) Y_2$
- $\nabla_{Y_2} Y_4 = 0$
- $\nabla_{Y_5} Y_2 = -\frac{1}{v}\cot(\theta) Y_1 - \frac{1}{2}\left(\frac{h}{h_r v} + \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_3$
- $\nabla_{Y_2} Y_5 = -\frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_3$
- $\nabla_{Y_5} Y_3 = \frac{1}{2}\left(\frac{h}{h_r v} + \frac{h_r}{hv} + \frac{v}{hh_r}\right) Y_2$
- $\nabla_{Y_3} Y_1 = \pm \frac{1}{h_r}\cot(\theta) Y_2 + \frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{vh} - \frac{v}{hh_r}\right) Y_4$
- $\nabla_{Y_5} Y_4 = \frac{1}{v}\cot(\theta) Y_5$
- $\nabla_{Y_3} Y_2 = \mp \frac{1}{h_r}\cot(\theta) Y_1 + \frac{1}{2}\left(\frac{h}{h_r v} - \frac{h_r}{hv} - \frac{v}{hh_r}\right) Y_5$
- $\nabla_{Y_3} Y_3 = -\frac{h_r'}{h_r} Y_6$
- $\nabla_{Y_5} Y_5 = -\frac{1}{v}\cot(\theta) Y_4 - \frac{v'}{v} Y_6$
- $\nabla_{Y_1} Y_6 = \frac{h'}{h} Y_1$ • $\nabla_{Y_2} Y_6 = \frac{h'}{h} Y_2$ • $\nabla_{Y_3} Y_6 = \frac{h_r'}{h_r} Y_3$ • $\nabla_{Y_4} Y_6 = \frac{v'}{v} Y_4$ • $\nabla_{Y_5} Y_6 = \frac{v'}{v} Y_5$
- $0 = \nabla_{Y_6} Y_1 = \nabla_{Y_6} Y_2 = \nabla_{Y_6} Y_3 = \nabla_{Y_6} Y_4 = \nabla_{Y_6} Y_5 = \nabla_{Y_6} Y_6$

By combining Theorem 3.3 with equation (1.2), and remembering that $Y_6 = X_6 = \frac{\partial}{\partial r}$ and $X_4 = \frac{\partial}{\partial \theta}$, we compute the following formulas for the $(4,0)$ curvature tensor R_μ. As in equations (3.5) through (3.8) we use the notation:

$$R^\mu_{ijkl} := \langle R_\mu(Y_i, Y_j)Y_k, Y_l\rangle_\mu.$$

Theorem 3.4. *In terms of the basis given in equation* (3.4)*, the only independent nonzero components of the* $(4,0)$ *curvature tensor* R_μ *are the following:*

$$R^\mu_{1212} = -\left(\frac{h'}{h}\right)^2 - \frac{1}{h^2} \qquad R^\mu_{4545} = -\left(\frac{v'}{v}\right)^2 + \frac{1}{v^2} \qquad R^\mu_{1515} = R^\mu_{2424} = -\frac{h'v'}{hv}$$

$$R^\mu_{1414} = R^\mu_{2525} = -\frac{v'h'}{vh} - \left(\frac{-v^2}{4h^2h_r^2} - \frac{h^2}{4v^2h_r^2} + \frac{3h_r^2}{4v^2h^2} - \frac{1}{2v^2} + \frac{1}{2h^2} - \frac{1}{2h_r^2}\right)$$

$$R^\mu_{3434} = R^\mu_{3535} = -\frac{v'h_r'}{vh_r} - \left(\frac{-v^2}{4h^2h_r^2} + \frac{3h^2}{4v^2h_r^2} - \frac{h_r^2}{4v^2h^2} - \frac{1}{2v^2} - \frac{1}{2h^2} + \frac{1}{2h_r^2}\right)$$

$$R^\mu_{1313} = R^\mu_{2323} = -\frac{h'h_r'}{hh_r} - \left(\frac{3v^2}{4h^2h_r^2} - \frac{h^2}{4v^2h_r^2} - \frac{h_r^2}{4v^2h^2} + \frac{1}{2v^2} + \frac{1}{2h^2} + \frac{1}{2h_r^2}\right)$$

$$R^\mu_{1616} = R^\mu_{2626} = -\frac{h''}{h} \qquad R^\mu_{3636} = -\frac{h_r''}{h_r} \qquad R^\mu_{4646} = R^\mu_{5656} = -\frac{v''}{v}$$

$$R^\mu_{1436} = R^\mu_{2536} = \frac{1}{2h_r}\left[\left(\frac{h}{v}\right)' - \left(\frac{v}{h}\right)' - \left(\frac{h_r^2}{vh}\right)'\right]$$

$$R^\mu_{1634} = R^\mu_{2635} = \frac{1}{2h}\left[-\left(\frac{h_r}{v}\right)' + \left(\frac{v}{h_r}\right)' + \left(\frac{h^2}{vh_r}\right)'\right]$$

$$R^\mu_{1346} = R^\mu_{2356} = \frac{-1}{2v}\left[\left(\frac{h}{h_r}\right)' + \left(\frac{h_r}{h}\right)' + \left(\frac{v^2}{hh_r}\right)'\right]$$

$$R^\mu_{1425} = \frac{1}{2v^2} - \frac{1}{2h^2} - \frac{h_r^2}{2h^2v^2}$$

$$R^\mu_{1245} = -\frac{1}{4}\left(\frac{h^2}{h_r^2v^2} + \frac{h_r^2}{h^2v^2} + \frac{v^2}{h^2h_r^2} + \frac{2}{h^2} + \frac{2}{h_r^2} - \frac{2}{v^2}\right)$$

$$R^\mu_{1524} = -\frac{1}{4}\left(\frac{-h^2}{h_r^2v^2} + \frac{h_r^2}{h^2v^2} - \frac{v^2}{h^2h_r^2} - \frac{2}{h_r^2}\right).$$

It is a tedious exercise in hyperbolic trigonometric identities to check that, when $h = \cosh(r)$, $h_r = \cosh(2r)$, and $v = \sinh(r)$, the above formulas reduce to the constants in equations (3.5) through (3.8). Also, note that the first nine equations above give the sectional curvatures of the coordinate planes, while the last six equations are formulas for the nonzero mixed terms.

Finally, notice that the above curvature formulas contain all of the formulas that arise in the analogous $\mathbb{CH}^n \setminus \mathbb{H}^n$ for general n. In general, one can write the complex hyperbolic metric $\mathbf{c_n}$ as

$$\mathbf{c_n} = \cosh^2(r)\mathbf{h_{n-1}} + \cosh^2(2r)dX_n^2 + \sinh^2(r)\sigma_{\mathbf{n-1}} + dr^2$$

and the corresponding warped-product metric as

$$\mu_{\mathbf{n}} = h^2(r)\mathbf{h}_{\mathbf{n-1}} + h_r^2(r)dX_n^2 + v^2(r)\sigma_{\mathbf{n-1}} + dr^2$$

where $\sigma_{\mathbf{n-1}}$ is the round metric on $\mathbb{S}^{n-1}$ and the vector field X_n is defined in the same manner as X_3. The curvature formulas for the base $\mathbb{H}^n$ are encoded in the formulas for R^μ_{1212} and R^μ_{1313}. All curvature formulas for the $\mathbb{S}^{n-1}$ factor are contained in the term R^μ_{4545}. Note that neither of these cases contain any mixed terms. Adding in the curvature formulas above of the form R^μ_{1414}, R^μ_{3434}, R^μ_{1425}, R^μ_{1245}, and R^μ_{1524} gives all curvature formulas for $h^2(r)\mathbf{h}_{\mathbf{n-1}}+h_r^2(r)dX_n^2+v^2(r)\sigma_{\mathbf{n-1}}$ (this is where most of the mixed terms appear). And then all of the formulas above containing a "6" give the rest of the curvature formulas for μ_n.

4. Curvature formulas for warped product metrics on $\mathbb{CH}^n \setminus \mathbb{CH}^k$

As mentioned in the Introduction, for simplicity we are going to restrict ourselves to the case when $n = 5$ and $k = 2$. These are the smallest choices for n and k which capture every formula for the curvature tensor in the general case, so nothing is lost with this restriction (see the comments after Theorem 4.3 for more discussion).

4.1. **Expressing the metric in $\mathbb{CH}^5$ in spherical coordinates about $\mathbb{CH}^2$.**

Let $\mathbf{c_5}$ denote the complex hyperbolic metric on $\mathbb{CH}^5$ normalized to have constant holomorphic sectional curvature -4. Since $\mathbb{CH}^2$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{CH}^5$, there exists an orthogonal projection map $\pi : \mathbb{CH}^5 \to \mathbb{CH}^2$. This map π is a fiber bundle whose fibers are totally geodesic 6-planes isometric to $\mathbb{CH}^3$.

For $r > 0$ let $E(r)$ denote the r-neighborhood of $\mathbb{CH}^2$. Then $E(r)$ is a real hypersurface in $\mathbb{CH}^5$, and consequently we can decompose $\mathbf{c_5}$ as

$$\mathbf{c_5} = (\mathbf{c_5})_r + dr^2$$

where $(\mathbf{c_5})_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r : E(r) \to \mathbb{CH}^2$ denote the restriction of π to $E(r)$. Note that π_r is an $\mathbb{S}^5$-bundle whose fiber over any point $q \in \mathbb{CH}^2$ is (topologically) the 5-sphere of radius r in the totally geodesic 6-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the 5-sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$. Note that this copy of $\mathbb{S}^5$ does *not* have constant sectional curvature equal to 1, but rather it is an example of a *Berger sphere*. This will be discussed further below.

For $r, s > 0$ there exists a diffeomorphism $\phi_{sr} : E(s) \to E(r)$ induced by the geodesic flow along the totally geodesic 6-planes orthogonal to $\mathbb{CH}^2$. Fix $p \in E(r)$ arbitrary, let $q = \pi(p) \in \mathbb{CH}^2$, and let γ be the unit speed geodesic such that $\gamma(0) = q$ and $\gamma(r) = p$. In what follows, all computations are considered in the tangent space $T_pE(r)$.

Note that $\mathcal{V}(r)$ is tangent to both $E(r)$ and the totally geodesic 6-plane $\pi^{-1}(q)$. Then since $\pi^{-1}(q)$ is preserved by the geodesic flow, we have that $d\phi_{sr}$ takes $\mathcal{V}(s)$ to $\mathcal{V}(r)$. Consider the complex geodesic $P = exp(span(\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}))$. P intersects $E(r)$ orthogonally, and $P \cap E(r)$ is isometric to a circle of radius r. Thus, since a complex geodesic has curvature -4, there exists a suitable identification $P \cong \mathbb{S}^1 \times (0, \infty)$

where the metric $\mathbf{c_5}$ restricted to P can be written as

$$\frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2$$

where $d\theta^2$ denotes the round metric on the unit circle $\mathbb{S}^1$. Note that the presence of the "1/4" is to make the metric complete when extended to the core $\mathbb{CH}^2$.

Notice that $\frac{\partial}{\partial\theta}$ is a vector field on the five sphere $\mathbb{S}^5$ mentioned above. More generally, thinking of $\mathbb{S}^5$ as the unit sphere in $\mathbb{C}^3$ with respect to the usual Hermitian metric, there is an obvious free action of the circle $\mathbb{S}^1$ on $\mathbb{S}^5$. The unit tangent vector field with respect to this action corresponds to the vector field $\frac{\partial}{\partial\theta}$ above. This action fibers $\mathbb{S}^5$ over the complex projective plane $\mathbb{CP}^2$, and the Riemannian submersion metric on this fiber bundle is an example of a Berger sphere (see [FJ94] pg. 59 for more details). Let $\alpha(t)$ be a unit speed geodesic in $\mathbb{S}^5$ orthogonal to $J\frac{\partial}{\partial r}$ such that $\alpha(0) = p$. Then $\exp_p(\alpha'(0), \partial r)$ forms a totally real totally geodesic 2-plane in $\mathbb{CH}^5$. Thus the curvature of this 2-plane is -1. Since the direction of α orthogonal to $J\frac{\partial}{\partial r}$ was arbitrary, we can write the Riemannian metric $(\mathbf{c_5})_r$ restricted to $\mathcal{V}(r)$ as

$$\sinh^2(r)\mathbf{p_2} + \frac{1}{4}\sinh^2(2r)d\theta^2$$

where $\mathbf{p_2}$ denotes the complex projective metric on $\mathbb{CP}^2$.

Now let $\beta(t)$ be any unit speed geodesic in $\mathbb{CH}^2$ such that $\beta(0) = q$. Then $Q = exp(span(\beta'(0), \gamma'(0)))$ is a totally real totally geodesic submanifold of $\mathbb{CH}^5$, and thus $K(\beta', \gamma') = -1$. Therefore, the metric $\mathbf{c_5}$ restricted to Q can be written as $\cosh^2(r)dt^2 + dr^2$. But since γ was arbitrary, we can write the metric on the 5-dimensional submanifold determined by $\mathbb{CH}^2$ and $\frac{\partial}{\partial r}$ as $\cosh^2 \mathbf{c_2} + dr^2$. This leads to the following.

Theorem 4.1. *The complex hyperbolic manifold $\mathbb{CH}^5 \setminus \mathbb{CH}^2$ can be written as $E \times (0,\infty)$ where $E \cong \mathbb{CH}^2 \times \mathbb{S}^5$ equipped with the metric*

$$\mathbf{c_5} = \cosh^2(r)\mathbf{c_2} + \sinh^2(r)\mathbf{p_2} + \frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2. \tag{4.1}$$

4.2. **The warped product metric, orthonormal basis, and curvature formulas in $\mathbb{CH}^5 \setminus \mathbb{CH}^2$.** For some positive, increasing real-valued functions $h, v, v_r : (0,\infty) \to \mathbb{R}$ define the Riemannian metrics

$$\gamma_{\mathbf{r},\theta} = h^2(r)\mathbf{c_2} + v^2(r)\mathbf{p_2}$$

$$\gamma_{\mathbf{r}} := \gamma_{\mathbf{r},\theta} + \frac{1}{4}v_r^2(r)d\theta^2$$

and

$$\gamma := \gamma_{\mathbf{r}} + dr^2. \tag{4.2}$$

Of course, $\gamma = \mathbf{c_5}$ when $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$.

For the remainder of this Section, fix $p = (q_1, \bar{q}, r) \in \mathbb{CH}^2 \times \mathbb{S}^5 \times (0,\infty) \cong \mathbb{CH}^5 \setminus \mathbb{CH}^2$, and write $\bar{q} \in \mathbb{S}^5$ as (q_2, θ) where $q_2 \in \mathbb{CP}^2$ and $\theta \in \mathbb{S}^1$. Let $(\check{X}_1, \check{X}_2, \check{X}_3, \check{X}_4)$ be an orthonormal collection of vector fields near $q_1 \in \mathbb{CH}^2$ which satisfies:

(1) $[\check{X}_i, \check{X}_j]_{q_1} = 0$ for all $1 \le i, j \le 4$.
(2) $J\check{X}_2|_{q_1} = \check{X}_1|_{q_1}$ and $J\check{X}_4|_{q_1} = \check{X}_3|_{q_1}$.

Define an analogous collection of vector fields $(\check{X}_5, \check{X}_6, \check{X}_7, \check{X}_8)$ about $q_2 \in \mathbb{CP}^2$ so that $J\check{X}_6|_{q_2} = \check{X}_5|_{q_2}$, $J\check{X}_8|_{q_2} = \check{X}_7|_{q_2}$, and $[\check{X}_i, \check{X}_j]_{q_2} = 0$ for all $5 \leq i, j \leq 8$. Extend both collections to vector fields $(X_1, \ldots, X_8)$ near p. Lastly, let $X_9 = \frac{\partial}{\partial\theta}$ and $X_{10} = \frac{\partial}{\partial r}$.

Define an orthonormal basis $\{Y_i\}_{i=1}^{8}$ with respect to γ by

(4.3)
$$\begin{array}{lllll} Y_1 = \dfrac{1}{h}X_1 & Y_2 = \dfrac{1}{h}X_2 & Y_3 = \dfrac{1}{h}X_3 & Y_4 = \dfrac{1}{h}X_4 & Y_5 = \dfrac{1}{v}X_5 \\ Y_6 = \dfrac{1}{v}X_6 & Y_7 = \dfrac{1}{v}X_7 & Y_8 = \dfrac{1}{v}X_8 & Y_9 = \dfrac{1}{\frac{1}{2}v_r}X_9 & Y_{10} = X_{10}. \end{array}$$

Our goal is to compute formulas for the components of the $(4,0)$ curvature tensor R_γ in terms of the warping functions h, v, and v_r. As a first step, we need to compute the components of the $(4,0)$ curvature tensor $R_{\mathbf{c_5}}$ of the complex hyperbolic metric with respect to the orthonormal basis given above. Just as in Section 3 we can do this with the help of formula (5.1). To use this formula note that, by construction, we have that $JY_2 = Y_1$, $JY_4 = Y_3$, $JY_6 = Y_5$, $JY_8 = Y_7$, and $JY_{10} = Y_9$ at the point p (and again, when the metric is $\mathbf{c_5}$. So, when $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$). Lastly, we use the notation

$$R^{\mathbf{c_5}}_{ijkl} := \langle R_{\mathbf{c_5}}(Y_i, Y_j)Y_k, Y_l\rangle_{\mathbf{c_5}}.$$

Then, up to the symmetries of the curvature tensor, the nonzero components of the $(4,0)$ curvature tensor $R_{\mathbf{c_5}}$ are

$$-4 = R^{\mathbf{c_5}}_{1212} = R^{\mathbf{c_5}}_{3434} = R^{\mathbf{c_5}}_{5656} = R^{\mathbf{c_5}}_{7878} = R^{\mathbf{c_5}}_{9,10,9,10} \tag{4.4}$$
$$-1 = R^{\mathbf{c_5}}_{ijij} \text{ where } \{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\}\} \tag{4.5}$$
$$-2 = R^{\mathbf{c_5}}_{ijkl} \text{ where } (i,j) \neq (k,l) \in \{(1,2),(3,4),(5,6),(7,8),(9,10)\} \tag{4.6}$$
$$-1 = R^{\mathbf{c_5}}_{ikjl} \text{ where } (i,j) \neq (k,l) \in \{(1,2),(3,4),(5,6),(7,8),(9,10)\} \tag{4.7}$$
$$1 = R^{\mathbf{c_5}}_{iljk} \text{ where } (i,j) \neq (k,l) \in \{(1,2),(3,4),(5,6),(7,8),(9,10)\} \tag{4.8}$$

Let us quickly note that, since $\mathbb{CP}^2$ is dual to $\mathbb{CH}^2$, we have the following curvature formulas for $R_{\mathbf{p_2}}$:

$$\begin{aligned} 4 &= R^{\mathbf{p_2}}_{5656} = R^{\mathbf{p_2}}_{7878} \\ 1 &= R^{\mathbf{p_2}}_{5757} = R^{\mathbf{p_2}}_{5858} = R^{\mathbf{p_2}}_{6767} = R^{\mathbf{p_2}}_{6868} \\ 2 &= R^{\mathbf{p_2}}_{5678} = 2R^{\mathbf{p_2}}_{5768} = -2R^{\mathbf{p_2}}_{5867}. \end{aligned}$$

In the above formulas, $R^{\mathbf{p_2}}_{ijkl} := \langle R_{\mathbf{p_2}}(Y_i, Y_j)Y_k, Y_l\rangle_{\mathbf{p_2}}$ and with the abuse of notation of Y_i denoting the restriction of Y_i to $\mathbb{CP}^2$.

4.3. **Lie brackets and curvature formulas for $\gamma_{\mathbf{r},\theta}$.** The vector fields $\{X_i\}_{i=1}^{10}$ form an orthogonal frame near p which satisfies the following properties (at p):

(1) $[X_i, X_j]$ is tangent to the level surfaces of r for $1 \leq i, j \leq 9$.
(2) $[X_i, X_j]$ is tangent to $\mathbb{CH}^2 \times \mathbb{S}^1$ for $1 \leq i, j \leq 4$, where $\mathbb{S}^1 \cong \exp_p(J\frac{\partial}{\partial r})$.
(3) $[X_i, X_j]$ is tangent to $\mathbb{S}^5$ for $5 \leq i, j \leq 8$.
(4) $[X_i, X_{10}] = 0$ since X_i is invariant under the flow of $\frac{\partial}{\partial r}$ for $1 \leq i \leq 9$.
(5) $[X_i, X_9] = 0$ since X_i is invariant under the flow of $\frac{\partial}{\partial\theta}$ for $1 \leq i \leq 8$.

(6) $[X_i, X_j] = 0$ for $i \in \{1,2,3,4\}$ and $j \in \{5,6,7,8\}$ since these vector fields were defined via inclusion.

By the above points, and since $[\check{X}_i, \check{X}_j]_p = 0$ for all $1 \leq i, j \leq 8$, there exist *structure constants* c_{ij} such that $[X_i, X_j]_p = c_{ij} X_9$. Note that $c_{ij} = -c_{ji}$. The following Lemma provides the values for the structure constants.

Lemma 4.2. *The values for the structure constants are $c_{12} = c_{34} = c_{56} = c_{78} = 2$, and all other (independent) structure constants are equal to zero.*

A quick note is that by "independent" structure constants we just mean that, obviously, $c_{21} = c_{43} = c_{65} = c_{87} = -2 \neq 0$.

Proof. All of the structure constants can be found by combining formula (5.7) with the curvature formulas (4.6) through (4.8). To see that $c_{12} = 2$, we combine equation (4.6) with (5.7) to obtain

$$\begin{aligned} 4 = 2R^{\mathbf{c_5}}_{10,9,1,2} &= 0 + 0 + \langle [Y_1, Y_2], Y_9 \rangle_{\mathbf{c_5}} \left(\ln \left[\frac{\frac{1}{4}\sinh^2(2r)}{\cosh^2(r)} \right] \right)' \\ &= \frac{c_{12}\sinh(2r)}{\cosh^2(r)} \left(\frac{\cosh(r)}{\sinh(r)} \right) = 2c_{12}. \end{aligned}$$

An analogous argument shows that $c_{34} = c_{56} = c_{78} = 2$. To see that $c_{13} = c_{14} = c_{23} = c_{24} = c_{57} = c_{58} = c_{67} = c_{68} = 0$ we use the same equations as above, but note that the left hand side is now 0 instead of 4.

Lastly, to see that $c_{15} = 0$, note that

$$\begin{aligned} 0 = R^{\mathbf{c_5}}_{10,1,5,9} &= 0 + \langle [Y_5, Y_1], Y_9 \rangle_{\mathbf{c_5}} \left(\ln \left[\frac{\frac{1}{2}\sinh(2r)}{\sinh(r)} \right] \right)' + 0 \\ &= \frac{-\frac{1}{2}c_{15}\sinh(2r)}{\sinh(r)\cosh(r)} (\ln 2\cosh(r))' = -c_{15}\tanh(r). \end{aligned}$$

The argument that the remaining structure constants are 0 is identical to the argument above. $\square$

We now, for some fixed r and θ, compute the components of the $(4,0)$ curvature tensor $R_{\lambda_{\mathbf{r},\theta}}$ with respect to the orthonormal frame $\{Y_i\}_{i=1}^8$. Since $[X_i, X_j] = 0$ for $i \in \{1,2,3,4\}$ and $j \in \{5,6,7,8\}$, the metric $\gamma_{\mathbf{r},\theta}$ is a product metric. Then since the $(4,0)$ curvature tensor scales like the metric, up to the symmetries of the curvature tensor the only nonzero components of $R_{\gamma_{\mathbf{r},\theta}}$ are

$$\begin{aligned} &R^{\gamma_{\mathbf{r},\theta}}_{1212} = R^{\gamma_{\mathbf{r},\theta}}_{3434} = -\frac{4}{h^2} && R^{\gamma_{\mathbf{r},\theta}}_{1313} = R^{\gamma_{\mathbf{r},\theta}}_{1414} = R^{\gamma_{\mathbf{r},\theta}}_{2323} = R^{\gamma_{\mathbf{r},\theta}}_{2424} = -\frac{1}{h^2} \\ &R^{\gamma_{\mathbf{r},\theta}}_{5656} = R^{\gamma_{\mathbf{r},\theta}}_{7878} = \frac{4}{v^2} && R^{\gamma_{\mathbf{r},\theta}}_{5757} = R^{\gamma_{\mathbf{r},\theta}}_{5858} = R^{\gamma_{\mathbf{r},\theta}}_{6767} = R^{\gamma_{\mathbf{r},\theta}}_{6868} = \frac{1}{v^2} \\ &R^{\gamma_{\mathbf{r},\theta}}_{1234} = 2R^{\gamma_{\mathbf{r},\theta}}_{1324} = -2R^{\gamma_{\mathbf{r},\theta}}_{1423} = -\frac{2}{h^2} && R^{\gamma_{\mathbf{r},\theta}}_{5678} = 2R^{\gamma_{\mathbf{r},\theta}}_{5768} = -2R^{\gamma_{\mathbf{r},\theta}}_{5867} = \frac{2}{v^2}. \end{aligned}$$

In particular, note that mixed terms of the form $R^{\gamma_{\mathbf{r},\theta}}_{1256}$ are 0.

4.4. **Curvature formulas for $\gamma_{\mathbf{r}}$.** Formulas (5.4) through (5.7) allow us to compute the $(4,0)$ curvature tensor R_γ in terms of R_{γ_r}. We use a very different approach from Section 3 to compute the nonzero components of R_{γ_r}. The background for our current computations can be found in [Bes87] pg. 235-242. The metric γ_r is a Riemannian submersion metric with (horizontal) base $\gamma_{r,\theta}$ and (vertical) fiber $\frac{1}{4}v_r^2 d\theta^2$. So our approach is to compute the A and T tensors of γ_r, and to then use Theorem 9.28 in [Bes87] to compute the components of R_{γ_r}.

First, the T tensor is identically zero since the vertical $\mathbb{S}^1$-fibers are totally geodesic. The argument for this is identical to that in Section 6 of [Bel11].

We now compute the A tensor associated with γ_r. By ([Bes87] Prop. 9.24) we have that

$$A_{X_1}X_2 = \frac{1}{2}\mathscr{V}[X_1, X_2] = X_9.$$

Analogously, $A_{X_3}X_4 = A_{X_5}X_6 = A_{X_7}X_8 = X_9$ and $A_{X_i}X_j = 0$ if $\{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$. Also, by ([Bes87] eqn. 9.21a) we have $A_{X_9}X_i = 0$ for $1 \leq i \leq 8$.

Now, by ([Bes87] eqn. 9.21d) we see that

$$\langle A_{X_1}X_9, X_2\rangle_{\gamma_r} = -\langle A_{X_1}X_2, X_9\rangle_{\gamma_r} = -\frac{1}{4}v_r^2.$$

By this same equation we know that there are no other nonzero components of $A_{X_1}X_9$. Therefore, $A_{X_1}X_9 = -\frac{1}{4}\frac{v_r^2}{h^2}X_2$. Analogously, we have that

$$A_{X_2}X_9 = \frac{1}{4}\frac{v_r^2}{h^2}X_1 \qquad A_{X_3}X_9 = -\frac{1}{4}\frac{v_r^2}{h^2}X_4 \qquad A_{X_4}X_9 = \frac{1}{4}\frac{v_r^2}{h^2}X_3$$

$$A_{X_5}X_9 = -\frac{1}{4}\frac{v_r^2}{v^2}X_6 \quad A_{X_6}X_9 = \frac{1}{4}\frac{v_r^2}{v^2}X_5 \quad A_{X_7}X_9 = -\frac{1}{4}\frac{v_r^2}{v^2}X_8 \quad A_{X_8}X_9 = \frac{1}{4}\frac{v_r^2}{v^2}X_7$$

We are now ready to use Theorem 9.28 from [Bes87] to compute the nonzero components of R_{γ_r}. By (9.28c) we have that

$$\langle R_{\gamma_r}(X_1, X_9)X_1, X_9\rangle_{\gamma_r} = \langle A_{X_1}X_9, A_{X_1}X_9\rangle_{\gamma_r} = \frac{1}{16}\frac{v_r^4}{h^2}$$

and thus

$$R^{\gamma_{\mathbf{r}}}_{1919} = \frac{4}{h^2 v_r^2}\langle R_{\gamma_r}(X_1, X_9)X_1, X_9\rangle_{\gamma_r} = \frac{v_r^2}{4h^4}. \tag{4.9}$$

Identically, $R^{\gamma_{\mathbf{r}}}_{2929} = R^{\gamma_{\mathbf{r}}}_{3939} = R^{\gamma_{\mathbf{r}}}_{4949} = \frac{v_r^2}{4h^4}$. Also, a completely analogous computation shows that

$$R^{\gamma_{\mathbf{r}}}_{5959} = R^{\gamma_{\mathbf{r}}}_{6969} = R^{\gamma_{\mathbf{r}}}_{7979} = R^{\gamma_{\mathbf{r}}}_{8989} = \frac{v_r^2}{4v^4}. \tag{4.10}$$

By (9.28f) we have that

$$\begin{aligned}\langle R_{\gamma_r}(X_1, X_2)X_1, X_2\rangle_{\gamma_r} &= \langle R_{\gamma_{r,\theta}}(X_1, X_2)X_1, X_2\rangle_{\gamma_{r,\theta}} - 3\langle A_{X_1}X_2, A_{X_1}X_2\rangle_{\gamma_r} \\ &= \langle R_{\gamma_{r,\theta}}(X_1, X_2)X_1, X_2\rangle_{\gamma_{r,\theta}} - \frac{3}{4}v_r^2\end{aligned}$$

and thus

$$R^{\gamma_r}_{1212} = R^{\gamma_{r,\theta}}_{1212} - \frac{3v_r^2}{4h^4} = -\frac{4}{h^2} - \frac{3v_r^2}{4h^4} = R^{\gamma_r}_{3434}. \tag{4.11}$$

An identical argument shows that

$$R^{\gamma_r}_{5656} = R^{\gamma_r}_{7878} = \frac{4}{v^2} - \frac{3v_r^2}{4v^4}. \tag{4.12}$$

Since $A_{X_i}X_j = 0$ if $\{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ the above argument also provides

$$R^{\gamma_r}_{ijij} = R^{\gamma_{r,\theta}}_{ijij} \qquad \text{if} \quad \{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}. \tag{4.13}$$

We now compute the mixed terms of R_{γ_r}.

$$\begin{aligned}\langle R_{\gamma_r}(X_1,X_2)X_3,X_4\rangle_{\gamma_r} &= \langle R_{\gamma_{r,\theta}}(X_1,X_2)X_3,X_4\rangle_{\gamma_{r,\theta}} - 2\langle A_{X_1}X_2, A_{X_3}X_4\rangle_{\gamma_r} \\ &= \langle R_{\gamma_{r,\theta}}(X_1,X_2)X_3,X_4\rangle_{\gamma_{r,\theta}} - \frac{1}{2}v_r^2\end{aligned}$$

and therefore

$$R^{\gamma_r}_{1234} = R^{\gamma_{r,\theta}}_{1234} - \frac{v_r^2}{2h^4} = -\frac{2}{h^2} - \frac{v_r^2}{2h^4} = 2R^{\gamma_r}_{1324} = -2R^{\gamma_r}_{1423}. \tag{4.14}$$

Identically

$$R^{\gamma_r}_{5678} = R^{\gamma_{r,\theta}}_{5678} - \frac{v_r^2}{2v^4} = \frac{2}{v^2} - \frac{v_r^2}{2v^4} = 2R^{\gamma_r}_{5768} = -2R^{\gamma_r}_{5867}. \tag{4.15}$$

Now

$$\langle R_{\gamma_r}(X_1,X_2)X_5,X_6\rangle_{\gamma_r} = \langle R_{\gamma_{r,\theta}}(X_1,X_2)X_5,X_6\rangle_{\gamma_{r,\theta}} - 2\langle A_{X_1}X_2, A_{X_5}X_6\rangle_{\gamma_r} = -\frac{1}{2}v_r^2$$

and hence

$$R^{\gamma_r}_{1256} = -\frac{v_r^2}{2h^2v^2} = R^{\gamma_r}_{1278} = R^{\gamma_r}_{3456} = R^{\gamma_r}_{3478}. \tag{4.16}$$

Finally, the same argument yields

$$R^{\gamma_r}_{1526} = R^{\gamma_r}_{1728} = R^{\gamma_r}_{3546} = R^{\gamma_r}_{3748} = -\frac{v_r^2}{4h^4v^4} \tag{4.17}$$

$$R^{\gamma_r}_{1625} = R^{\gamma_r}_{1827} = R^{\gamma_r}_{3645} = R^{\gamma_r}_{3847} = \frac{v_r^2}{4h^4v^4}. \tag{4.18}$$

4.5. **Curvature formulas for γ.** Combining equations (4.9) through (4.18) with formulas (5.4) through (5.7) proves the following Theorem.

Theorem 4.3. *In terms of the basis given in equation* (4.3)*, the only independent nonzero components of the* $(4,0)$ *curvature tensor* R_γ *are given by the following formulas, where* $i \in \{1,2,3,4\}$*,* $k \in \{5,6,7,8\}$*,* $(i,j) \in \{(1,2),(3,4)\}$*, and* $(k,l) \in \{(5,6),(7,8)\}$*.*

$$R^{\gamma}_{1212} = R^{\gamma}_{3434} = -\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3v_r^2}{4h^4} \qquad R^{\gamma}_{5656} = R^{\gamma}_{7878} = -\left(\frac{v'}{v}\right)^2 + \frac{4}{v^2} - \frac{3v_r^2}{4v^4}$$

$$R^{\gamma}_{i9i9} = -\frac{h'v_r'}{hv_r} + \frac{v_r^2}{4h^4} \qquad R^{\gamma}_{k9k9} = -\frac{v'v_r'}{vv_r} + \frac{v_r^2}{4v^4} \qquad R^{\gamma}_{ikik} = -\frac{h'v'}{hv}$$

$$R^{\gamma}_{1313} = R^{\gamma}_{1414} = R^{\gamma}_{2323} = R^{\gamma}_{2424} = -\left(\frac{h'}{h}\right)^2 - \frac{1}{h^2}$$

(continued on next page)

$$R^\gamma_{5757} = R^\gamma_{5858} = R^\gamma_{6767} = R^\gamma_{6868} = -\left(\frac{v'}{v}\right)^2 + \frac{1}{v^2}$$

$$R^\gamma_{i,10,i,10} = -\frac{h''}{h} \qquad R^\gamma_{k,10,k,10} = -\frac{v''}{v} \qquad R^\gamma_{9,10,9,10} = -\frac{v_r''}{v_r}$$

$$R^\gamma_{1234} = 2R^\gamma_{1324} = -2R^\gamma_{1423} = -\frac{2}{h^2} - \frac{v_r^2}{2h^4}$$

$$R^\gamma_{5678} = 2R^\gamma_{5768} = -2R^\gamma_{5867} = \frac{2}{v^2} - \frac{v_r^2}{2v^4}$$

$$R^\gamma_{ijkl} = 2R^\gamma_{ikjl} = -2R^\gamma_{iljk} = -\frac{v_r^2}{2h^2v^2}$$

$$R^\gamma_{i,j,9,10} = 2R^\gamma_{i,9,j,10} = -2R^\gamma_{i,10,j,9} = -\frac{v_r}{h^2}\left(\ln\frac{v_r}{h}\right)'$$

$$R^\gamma_{k,l,9,10} = 2R^\gamma_{k,9,l,10} = -2R^\gamma_{k,10,l,9} = -\frac{v_r}{v^2}\left(\ln\frac{v_r}{v}\right)'.$$

Unlike Section 3, this time is is a much simpler exercise in hyperbolic trigonometric identities to check that, when $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$, the above formulas reduce to the constants in equations (4.4) through (4.8).

Finally, notice that the above curvature formulas contain all of the formulas that arise in the analogous $\mathbb{CH}^n \setminus \mathbb{CH}^k$ for general n and k. In general, one can write the complex hyperbolic metric $\mathbf{c_n}$ as

$$\mathbf{c_n} = \cosh^2(r)\mathbf{c_k} + \sinh^2(r)\mathbf{p_{n-k-1}} + \frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2$$

and the corresponding warped-product metric as

$$\gamma_\mathbf{n} = h^2(r)\mathbf{c_k} + v^2(r)\mathbf{p_{n-k-1}} + \frac{1}{4}v_r^2(r)d\theta^2 + dr^2$$

where $\mathbf{p_{n-k-1}}$ is the complex projective metric on $\mathbb{CP}^{n-k-1}$. The curvature formulas for the base $\mathbb{CH}^k$ are encoded in the formulas for $R^\gamma_{1212}, R^\gamma_{1313}$, and R^γ_{1234}. The analogous formulas for $\mathbb{CP}^{n-k-1}$ are contained in $R^\gamma_{5656}, R^\gamma_{5757}$, and R^γ_{5678}. Adding in the curvature formulas above of the form R^γ_{ijij} and R^γ_{ijkl} gives all curvature formulas for $h^2(r)\mathbf{c_k} + v^2(r)\mathbf{p_{n-k-1}}$. And then all of the formulas above containing either a "9" or a "10" give the rest of the curvature formulas for $\gamma_\mathbf{n}$.

4.6. **The exceptional case $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2}$.** Notice that, when $k = n-2$, there are no sectional curvatures of γ of the form

$$-\left(\frac{v'}{v}\right)^2 + \frac{1}{v}^2.$$

That is because we can write $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2} \cong \mathbb{CH}^{n-2} \times \mathbb{S}^3 \times (0,\infty)$, and $\mathbb{CP}^1$ (the base of the Hopf fibration) has constant holomorphic curvature 4. So the purpose of this subsection is to prove the following:

Lemma 4.4. *There do not exist functions h, v, and v_r that, when inserted into equation* (4.2)*, yield a complete finite volume Riemannian metric on $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2}$ with nonpositive sectional curvature.*

Proof. Since we are removing a copy of $\mathbb{CH}^{n-2}$ from $\mathbb{CH}^n$, for the metric to be complete we need to define h, v, and v_r on $(-\infty, \infty)$. These functions need to be positive for the metric to be Riemannian, and they need to be non-decreasing for there to be any chance of nonpositive curvature. For any hope of finite volume, we need all of the following limits to hold:

$$\lim_{r\to-\infty} h, h', v, v', v_r, v_r' = 0.$$

Now, from the formula for the R_{5656} term in Theorem 4.3 we must have that

$$\begin{aligned} &\frac{4-(v')^2}{v^2} - \frac{3v_r^2}{4v^4} \leq 0 \\ \iff \qquad &\frac{16-4(v')^2}{3} \leq \left(\frac{v_r}{v}\right)^2. \end{aligned} \tag{4.19}$$

In particular, we see that a necessary requirement for nonpositive curvature is that

$$\lim_{r\to-\infty} \frac{v_r}{v} > 1. \tag{4.20}$$

From the formula for the R_{k9k9} term in Theorem 4.3 we must have that

$$\begin{aligned} &-\frac{v'v_r'}{vv_r} + \frac{v_r^2}{4v^4} \leq 0 \\ \implies \qquad &\frac{3v_r^2}{4v^4} \leq \frac{3v'v_r'}{vv_r}. \end{aligned} \tag{4.21}$$

Comparing equations (4.19) and (4.21), we see that

$$\frac{4-(v')^2}{v^2} \leq \frac{3v'v_r'}{vv_r} \qquad \implies \qquad 4-(v')^2 \leq (3v'v_r') \cdot \frac{v}{v_r}$$

is also a necessary requirement for nonpositive curvature. But as $r \to -\infty$, we know that $4-(v')^2 \to 4$ and $3v'v_r' \to 0$. Thus, we must have that

$$\lim_{r\to-\infty} \frac{v}{v_r} = \infty \qquad \implies \qquad \lim_{r\to-\infty} \frac{v_r}{v} = 0. \tag{4.22}$$

Equations (4.20) and (4.22) provide a contradiction, proving the Lemma.

□

5. Preliminaries

5.1. **Formula for the curvature tensor of $\mathbb{CH}^n$ in terms of the complex structure J.** The components of the (4,0) curvature tensor of the complex hyperbolic metric g can be expressed in terms of g and the complex structure J. The following formula can be found in [KN96] or in Section 5 of [Bel11] (recall Remark 1.2 from the Introduction). In this formula X, Y, Z, and W are arbitrary vector fields.

$$\begin{aligned} \langle R_g(X,Y)Z, W\rangle_g = {} & \langle X,W\rangle_g \langle Y,Z\rangle_g - \langle X,Z\rangle_g\langle Y,W\rangle_g \\ & + \langle X,JW\rangle_g\langle Y,JZ\rangle_g - \langle X,JZ\rangle_g\langle Y,JW\rangle_g + 2\langle X,JY\rangle_g\langle W,JZ\rangle_g. \end{aligned} \tag{5.1}$$

5.2. **Koszul's formula for the Levi-Civita connection.** Let X, Y, and Z denote vector fields on a Riemannian manifold (M, g). The following is the well-known "Koszul formula" for the values of the Levi-Civita connection ∇ (which can be found on pg. 55 of [doC92])

$$\begin{aligned}\langle \nabla_Y X, Z\rangle_g = \frac{1}{2}(X\langle Y, Z\rangle_g + Y\langle Z, X\rangle_g - Z\langle X, Y\rangle_g \\ - \langle [X, Z], Y\rangle_g - \langle [Y, Z], X\rangle_g - \langle [X, Y], Z\rangle_g).\end{aligned} \tag{5.2}$$

In this paper we will usually be considering an orthonormal frame (Y_i). In this setting we know that $\langle Y_i, Y_j\rangle_g = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta. Therefore the first three terms on the right hand side of formula (5.2) are all zero. Thus, in an orthonormal frame, formula (5.2) reduces to

$$\langle \nabla_Y X, Z\rangle_g = -\frac{1}{2}\left(\langle [X, Z], Y\rangle_g + \langle [Y, Z], X\rangle_g + \langle [X, Y], Z\rangle_g\right). \tag{5.3}$$

5.3. **General curvature formulas for warped product metrics.** The curvature formulas below, which were worked out by Belegradek in [Bel12] and stated in Appendix B of [Bel11], apply to metrics of the form $g = g_r + dr^2$ on manifolds of the form $E \times I$ where I is an open interval and E is a manifold. The formulas are true provided that for each point $q \in E$ there exists a local frame $\{X_i\}$ on a neighborhood U_q in E which is g_r-orthogonal for each r. Such a family of metrics (E, g_r) is called *simultaneously diagonalizable*. Let

$$h_i(r) := \sqrt{g_r(X_i, X_i)}.$$

Then the local frame $\{Y_i\}$ defined by

$$Y_i = \frac{1}{h_i} X_i$$

is a g_r-orthonormal frame on U_q for any value of r. We then have the following formulas for the (4,0) curvature tensor R_g in terms of the (4,0) curvature tensor R_{g_r}, the collection $\{h_i\}$, and the Lie brackets $[Y_i, Y_j]$. Note that $\langle , \rangle$ is used to denote the metric g and $\partial r = \frac{\partial}{\partial r}$.

$$\langle R_g(Y_i, Y_j)Y_i, Y_j\rangle = \langle R_{g_r}(Y_i, Y_j)Y_i, Y_j\rangle - \frac{h_i' h_j'}{h_i h_j} \tag{5.4}$$

$$\langle R_g(Y_i, Y_j)Y_k, Y_l\rangle = \langle R_{g_r}(Y_i, Y_j)Y_k, Y_l\rangle \quad \text{if } \{i, j\} \neq \{k, l\} \tag{5.5}$$

$$\langle R_g(Y_i, \partial r)Y_i, \partial r\rangle = -\frac{h_i''}{h_i} \qquad \langle R_g(Y_i, \partial r)Y_j, \partial r\rangle = 0 \quad \text{if } i \neq j \tag{5.6}$$

$$\begin{aligned}2\langle R(\partial r, Y_i) Y_j, Y_k\rangle = \langle [Y_i, Y_k], Y_j\rangle \left(\ln \frac{h_j}{h_k}\right)' + \langle [Y_j, Y_i], Y_k\rangle \left(\ln \frac{h_k}{h_j}\right)' \\ + \langle [Y_j, Y_k], Y_i\rangle \left(\ln \frac{h_i^2}{h_j h_k}\right)'.\end{aligned} \tag{5.7}$$

5.4. **The Nijenhuis Tensor.** In Sections 3 and 4 we will be explicitly dealing with $\mathbb{CH}^n$. Since the almost complex structure on $\mathbb{CH}^n$ is integrable, we have that the *Nijenhuis Tensor* is identically equal to zero. Explicitly, for any vector fields X and Y on $\mathbb{CH}^n$, we have that

$$0 = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \tag{5.8}$$

where J denotes the complex structure on $\mathbb{CH}^n$.

6. Computations for the Lie brackets for $\mathbb{CH}^3 \setminus \mathbb{H}^3$

The whole purpose of this Section is to prove Theorem 3.2.

Proof of Theorem 3.2. There are $5 \times 10 = 50$ structure constants to compute from equation (3.9). From equation (3.10) we know that $c_{45}^4 = 0$ and $c_{45}^5 = -\cot(\theta)$, leaving 48 unknown structure constants.

We can combine formula (5.7) with equations (3.5) through (3.8) to compute many of the constants. As a first example, note that

$$\begin{aligned} 0 = 2R^{\mathbf{c_3}}_{6131} &= 0 + 2\langle [Y_3, Y_1], Y_1\rangle_{\mathbf{c_3}} \left(\ln \frac{h}{h_r}\right)' \\ &= -\frac{2c_{13}^1}{\cosh(2r)}\left(\ln \frac{\cosh(r)}{\cosh(2r)}\right)' \end{aligned}$$

and thus $c_{13}^1 = 0$. We can analogously show

$$\begin{aligned} 0 &= c_{14}^1 = c_{15}^1 = c_{23}^2 = c_{24}^2 = c_{25}^2 = c_{13}^3 = c_{23}^3 \\ &= c_{34}^3 = c_{35}^3 = c_{14}^4 = c_{24}^4 = c_{34}^4 = c_{15}^5 = c_{25}^5 = c_{35}^5. \end{aligned}$$

This narrows us down to 32 unknown constants.

Continuing with the same formula and equations, we have that

$$\begin{aligned} 0 = 2R^{\mathbf{c_3}}_{6145} &= 0 + 0 + \langle [Y_4, Y_5], Y_1\rangle_{\mathbf{c_3}} \left(\ln \frac{\cosh^2(r)}{\sinh^2(r)}\right)' \\ &= \frac{c_{45}^1 h}{v^2}\left(\ln \frac{\cosh^2(r)}{\sinh^2(r)}\right)' \end{aligned}$$

and therefore $c_{45}^1 = 0$. Analogously, $c_{45}^2 = c_{45}^3 = c_{12}^3 = c_{12}^4 = c_{12}^5 = 0$. This reduces us to 26 unknowns. But we can also use the same curvature formulas here, but with the indices permuted, to derive some simple equations relating some of the constants. For example,

$$\begin{aligned} 0 = R^{\mathbf{c_3}}_{6415} &= 0 + \langle [Y_1, Y_4], Y_5\rangle_{\mathbf{c_3}} \left(\ln \frac{\sinh(r)}{\cosh(r)}\right)' + \langle [Y_1, Y_5], Y_4\rangle_{\mathbf{c_3}} \left(\ln \frac{\sinh(r)}{\cosh(r)}\right)' \\ &= \frac{1}{h}(c_{14}^5 + c_{15}^4)\left(\ln \frac{\sinh(r)}{\cosh(r)}\right)' \end{aligned}$$

and thus $c_{14}^5 = -c_{15}^4$. Analogously, we have the identities

$$c_{24}^5 = -c_{25}^4 \qquad c_{34}^5 = -c_{35}^4 \qquad c_{13}^2 = -c_{23}^1$$
$$c_{14}^2 = -c_{24}^1 \qquad c_{15}^2 = -c_{25}^1.$$

Combining formula (5.7) with the fact that $2R^{\mathbf{c_3}}_{6413} = 2$ gives that

$$\begin{aligned}
2 &= \langle [Y_4, Y_3], Y_1\rangle_{\mathbf{c_3}} \left(\ln \frac{h}{h_r}\right)' + \langle [Y_1, Y_4], Y_3\rangle_{\mathbf{c_3}} \left(\ln \frac{h_r}{h}\right)' + \langle [Y_1, Y_3], Y_4\rangle_{\mathbf{c_3}} \left(\ln \frac{v^2}{hh_r}\right)' \\
&= -\frac{c^1_{34} h}{h_r v}\left(\ln \frac{h}{h_r}\right)' + \frac{c^3_{14} h_r}{hv}\left(\ln \frac{h_r}{h}\right)' + \frac{c^4_{13} v}{hh_r}\left(\ln \frac{v^2}{hh_r}\right)' \\
&= \left(\frac{-c^1_{34}\cosh(r)}{\cosh(2r)\sinh(r)} - \frac{c^3_{14}\cosh(2r)}{\cosh(r)\sinh(r)}\right)(\tanh(r) - 2\tanh(2r)) \\
&+ \frac{c^4_{13}\sinh(r)}{\cosh(r)\cosh(2r)}\left(2\coth(r) - \tanh(r) - 2\tanh(2r)\right).
\end{aligned}$$

Now one consults equation (5.9) in [Min17] to see that the solutions to this equation are

$$c^4_{13} = 1 \qquad c^3_{14} = 1 \qquad c^1_{34} = -1.$$

In exactly the same manner we can use $R^{\mathbf{c_3}}_{6253}$ with equation (5.9) in [Min17] to compute

$$c^5_{23} = 1 \qquad c^3_{25} = 1 \qquad c^2_{35} = -1.$$

This leaves 20 unknowns together with the 6 identities listed above. Now, using $R^{\mathbf{c_3}}_{6135}$ we have that

$$\begin{aligned}
0 &= \langle [Y_1, Y_5], Y_3\rangle_{\mathbf{c_3}} \left(\ln \frac{h_r}{v}\right)' + \langle [Y_3, Y_1], Y_5\rangle_{\mathbf{c_3}} \left(\ln \frac{v}{h_r}\right)' + \langle [Y_3, Y_5], Y_1\rangle_{\mathbf{c_3}} \left(\ln \frac{h^2}{h_r v}\right)' \\
&= \left(\frac{c^3_{15}\cosh(2r)}{\cosh(r)\sinh(r)} + \frac{c^5_{13}\sinh(r)}{\cosh(r)\cosh(2r)}\right)(2\tanh(2r) - \coth(r)) \\
&+ \frac{c^1_{35}\cosh(r)}{\cosh(2r)\sinh(r)}\left(2\tanh(r) - 2\tanh(2r) - \coth(r)\right).
\end{aligned}$$

One can check that the only solution to this equation (which holds for all values of r) is $c^5_{13} = c^3_{15} = c^1_{35} = 0$. Analogously, we can use $\mathbb{R}^{\mathbf{c_3}}_{6234}$ to show that $c^4_{23} = c^3_{24} = c^2_{34} = 0$. These equations reduce us to 14 unknowns.

This is as much information as we can gain from formula (5.7). So we next turn to the Nijenhuis Tensor (5.8). First applying this to (Y_1, Y_2), we have

$$\begin{aligned}
0 &= [Y_1, Y_2] - J[Y_4, Y_2] - J[Y_1, Y_5] - [Y_4, Y_5] \\
&= \frac{1}{h}(c^1_{12} Y_1 + c^2_{12} Y_2) + J\left(\frac{c^1_{24}}{v} Y_1 + \frac{c^5_{24}}{h} Y_5\right) - J\left(\frac{c^2_{15}}{v} Y_2 + \frac{c^4_{15}}{h} Y_4\right) + \frac{1}{v}\cot(\theta) Y_5 \\
&= \frac{1}{h}\left(c^1_{12} - c^4_{15}\right) Y_1 + \frac{1}{h}(c^2_{12} + c^5_{24}) Y_2 - \frac{c^1_{24}}{v} Y_4 + \frac{1}{v}(c^2_{15} + \cot(\theta)) Y_5.
\end{aligned}$$

Therefore, we have that

$$c^1_{24} = 0 = -c^2_{14} \qquad c^2_{15} = -\cot(\theta) = -c^1_{25} \qquad c^1_{12} = c^4_{15} = -c^5_{14} \qquad c^2_{12} = -c^5_{24}.$$

We can also apply the Nijenhuis tensor to the pairs (Y_1, Y_3) and (Y_2, Y_3), but these are much less productive. These applications only give us the pair of identities

$$c^2_{13} = -c^5_{34} \qquad c^1_{23} = -c^4_{35}$$

the former of which comes from the pair (Y_1, Y_3), and the latter from the pair (Y_2, Y_3).

At this stage, we have reduced our 10 Lie brackets as follows:

$$
\begin{aligned}
&[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 && [X_1, X_3] = c_{13}^2 X_2 + X_4 \\
&[X_1, X_4] = X_3 - c_{12}^1 X_5 && [X_1, X_5] = -\cot(\theta) X_2 + c_{12}^1 X_4 \\
&[X_2, X_3] = -c_{13}^2 X_1 + X_5 && [X_2, X_4] = -c_{12}^2 X_5 \\
&[X_2, X_5] = \cot(\theta) X_1 + X_3 + c_{12}^2 X_4 && [X_3, X_4] = -X_1 - c_{13}^2 X_5 \\
&[X_3, X_5] = -X_2 + c_{13}^2 X_4 && [X_4, X_5] = -\cot(\theta) X_5.
\end{aligned}
$$

Notice that, using the known identities, we can reduce the system to three unknowns: c_{12}^1, c_{12}^2, and c_{13}^2. All that is left is to show that $c_{12}^1 = \pm 1$, $c_{12}^2 = 0$, and $c_{13}^2 = \mp \cot(\theta)$.

At this point we have exhausted all of our "easy" options. The only way to obtain new relationships between the structure constants is to compute new components of $R_{\mathbf{c_3}}$. To do this, one needs to first use equation (5.2) with the values for the Lie brackets given above to compute the Levi-Civita connection ∇ compatible with $\mathbf{c_3}$. Of course, these formulas will contain the constants c_{12}^1, c_{12}^2, and c_{13}^2. But when the correct values for these constants are inserted, these formulas will reduce to those of Theorem 3.3. Then once one has computed ∇, they can use those values to compute the components of $R_{\mathbf{c_3}}$.

The first component that will be useful is $R_{1212}^{\mathbf{c_3}}$:

$$
\begin{aligned}
-1 = & R_{1212}^{\mathbf{c_3}} = \langle \nabla_{Y_2} \nabla_{Y_1} Y_1 - \nabla_{Y_1} \nabla_{Y_2} Y_1 + \nabla_{[Y_1, Y_2]} Y_1, Y_2 \rangle_{\mathbf{c_3}} \\
= & \langle \nabla_{Y_2} \left(\frac{-c_{12}^1}{\cosh(r)} Y_2 - \tanh(r) Y_6 \right) - \nabla_{Y_1} \left(\frac{-c_{12}^2}{\cosh(r)} Y_2 \right) \\
& + \frac{c_{12}^1}{\cosh(r)} \nabla_{Y_1} Y_1 + \frac{c_{12}^2}{\cosh(r)} \nabla_{Y_2} Y_1, Y_1 \rangle_{\mathbf{c_3}} \\
= & -\frac{\sinh^2(r)}{\cosh^2(r)} - \frac{((c_{12}^1)^2 + (c_{12}^2)^2)}{\cosh^2(r)} \\
\implies & \qquad (c_{12}^1)^2 + (c_{12}^2)^2 = 1.
\end{aligned}
$$

The next component that we use is $R_{1512}^{\mathbf{c_3}}$. We will skip the details and just note that

$$
0 = R_{1512}^{\mathbf{c_3}} = \frac{c_{12}^2}{\sinh(r)\cosh(r)} \cdot \cot(\theta)
$$

which implies that $c_{12}^2 = 0$. Combining this with the first equation shows that $c_{12}^1 = \pm 1$. Finally, to compute c_{13}^2 we use $R_{1412}^{\mathbf{c_3}}$:

$$
0 = R_{1412}^{\mathbf{c_3}} = \frac{-c_{13}^2}{\sinh(r)\cosh(r)} - \frac{c_{12}^1}{\sinh(r)\cosh(r)} \cdot \cot(\theta).
$$

Therefore

$$
c_{13}^2 = -(\pm 1)\cot(\theta) = \mp \cot(\theta).
$$

□

Acknowledgements. The author would like to thank J.F. Lafont for both suggesting this problem and for many helpful discussions about the work contained in this paper. The author would also like to thank I. Belegradek, G. Avramidi, and T. N. Phan for various comments which aided in this research. Lastly, many of these conversations would not have been possible without support from an AMS-Simons travel grant.

References

[AP16] G. Avramidi and T.N. Phan, *Half dimensional collapse of ends of manifolds of non-positive curvature*, preprint, arxiv: 1608.02185.

[Bel11] I. Belegradek, *Complex hyperbolic hyperplane complements*, Math. Ann. **353** (2012), no. 2, 545-579.

[Bel12] I. Belegradek, *Rigidity and relative hyperbolicity of real hyperbolic hyperplane complements*, Pure Appl. Math. Q. **8** (2012), no. 1, 15-51.

[Bes87] A. L. Besse, *Einstein manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **10**, Springer-Verlag, Berlin (1987).

[doC92] M.P. do Carmo, *Riemannian Geometry*, Birkhäuser Boston (1992).

[FJ94] F.T. Farrell and L.E. Jones, *Complex hyperbolic manifolds and exotic smooth structures*, Invent. math. **117** (1994), no. 1, 57-74.

[GT87] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), 1-12.

[KN96] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. II, Wiley Classics Library, Wiley, New York (1996). Reprint of the 1969 original.

[LMMT] J.F. Lafont, J. Meyer, B. Minemyer, and B. Tshishiku, *New examples of negatively curved 4-manifolds*, preprint.

[Min17] B. Minemyer, *Real hyperbolic hyperplane complements in the complex hyperbolic plane*, submitted, arxiv: 1609.01974.

[Min18a] B. Minemyer, *Negatively curved codimension 1 distributions*, submitted, preliminary version available at "http://facstaff.bloomu.edu/bminemyer/Research.html".

[Min18b] B. Minemyer, *Complex hyperbolic Gromov-Thurston manifolds*, preprint, preliminary version available at "http://facstaff.bloomu.edu/bminemyer/Research.html".

[Ont15] P. Ontaneda, *On the Farrell and Jones warping deformation*, J. Lond. Math. Soc. **92** (2015), no. 3, 566-582.

Department of Mathematical and Digital Sciences, Bloomsburg University, Bloomsburg, Pennsylvania 17815

Email address: bminemyer@bloomu.edu