# NEGATIVELY CURVED CODIMENSION ONE DISTRIBUTIONS

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ABSTRACT. We consider finite volume manifold pairs $(M, N)$ modeled on $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$ and prove the existence of a special Riemannian metric $g$ on $M \setminus N$. This metric $g$ is complete, has finite volume, and is negatively curved when restricted to a specific nonintegrable codimension one distribution $\mathscr{D}$. The existence of this metric $g$ shows that some recent results in [AP16] cannot, in some sense, be extended to distributions on manifolds.

## 1. Introduction

Let $\mathbb{CH}^n$ denote (complex) $n$-dimensional complex hyperbolic space. If $M$ is a Riemannian manifold and $N$ a totally geodesic submanifold of $M$, we say that the pair $(M, N)$ is *modeled on* $(\mathbb{CH}^n, \mathbb{CH}^k)$ if there exist lattices $\Gamma \subset \text{Isom}(\mathbb{CH}^n)$ and $\Lambda \subset \text{Isom}(\mathbb{CH}^k)$ such that $M = \mathbb{CH}^n/\Gamma$, $N = \mathbb{CH}^k/\Lambda$, and $\Lambda < \Gamma$. We also allow for the possibility that $N$ is disconnected. That is, we allow for multiple lattices $\Lambda < \Gamma$ which correspond to different (disjoint) copies of $\mathbb{CH}^k \subset \mathbb{CH}^n$. The main result of this paper is the existence of a special metric on $M \setminus N$ when $k = n - 2$ (so, when $N$ has real codimension 4).

**Theorem 1.1.** *Suppose $(M, N)$ is modeled on $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$ with $M$ having finite volume. Then there exists a Riemannian metric $g$ on $M \setminus N$ which is complete, has finite volume, and such that $M$ admits a codimension one (nonintegrable) distribution $\mathscr{D}$ which satisfies*

$$K_g(\sigma) < -\delta$$

*for all 2-planes $\sigma \subseteq \mathscr{D}$ and for some $\delta > 0$.*

In the above Theorem, $K_g$ denotes the sectional curvature with respect to $g$.

To motivate Theorem 1.1, consider the case when $n = 3$, $M = \mathbb{CH}^3$, and $N$ is a single copy of $\mathbb{CH}^1$ (and for simplicity we ignore the assumption of finite volume). Here, $M \setminus N$ is a six dimensional manifold diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^3 \times (0, \infty)$, and $\mathscr{D}$ is a five dimensional distribution. A recent result of Avramidi and Phan [AP16] shows that the ends of a complete, nonpositively curved five dimensional manifold with finite volume must be aspherical. From the construction of the distribution $\mathscr{D}$ it is clear that, if it were integrable, the ends of the corresponding submanifold would have nontrivial $\Pi_2$ and thus would not be aspherical (thinking of the copy of $\mathbb{S}^3$ as the total space of the Hopf fibration over $\mathbb{S}^2$, the one direction not included in $\mathscr{D}$ is the direction tangent to the $\mathbb{S}^1$ fiber at each point of the base $\mathbb{CP}^1$). So Thereom 1.1 shows that the results of [AP16] cannot, in some sense, be extended to include nonintegrable distributions in manifolds.

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The proof of Theorem 1.1 is technical, but some of the methods will likely be more useful than the result. All calculations are performed in the universal cover $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$, but will clearly "descend" to the pair $(M, N)$. To prove Theorem 1.1 we first write the metric in $\mathbb{CH}^n$ in spherical coordinates about a copy of $\mathbb{CH}^{n-2}$ (Theorem 2.1). We then consider the corresponding warped-product metric $\gamma$ (equation (2.2)) and calculate formulas for the components of the $(4, 0)$ sectional curvature tensor of $\gamma$ (Theorem 2.2). All of this is contained in Section 2, and is really just a special case of some of the results in [Min18]. But we include the necessary details here for the sake of exposition.

In Section 3 we discuss which direction we omit for the distribution $\mathscr{D}$ and develop general curvature formulas to prove Theorem 1.1 (equations (3.3) and (3.4)). Finally, in Section 4 we prove the existence of warping functions which lead to a complete, finite volume metric whose sectional curvature is bounded above by a negative constant when restricted to 2-planes contained in $\mathscr{D}$. Some of the methods used in this argument are similar to those used in [GT87], [Bel12], [Bel11], and [Min17].

*Remark* 1.2. In this paper we scale the complex hyperbolic metric to have sectional curvatures in the interval $[-4, -1]$. We also follow the notation of [doC92] and use the following formula for the curvature tensor $R$ of $g$

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

This coincides with the notation used in [Min18], but varies from that used in some of the references in the preceding paragraph.

## 2. Curvature formulas for warped product metrics on $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2}$

In this Section we quickly describe how to write the metric in $\mathbb{CH}^n$ in spherical coordinates about a totally geodesic copy of $\mathbb{CH}^{n-2}$. We then describe the corresponding warped product metric, and state the formulas for the components of the $(4, 0)$ sectional curvature tensor with respect to this metric. All calculations for the formulas found in this Section can be found in [Min18] and, in particular, all formulas in this Section are a special case of those worked out in Section 4 of [Min18].

### 2.1. Expressing the metric in $\mathbb{CH}^n$ in spherical coordinates about $\mathbb{CH}^{n-2}$.

Let $\mathbf{c_n}$ denote the complex hyperbolic metric on $\mathbb{CH}^n$ normalized to have constant holomorphic sectional curvature $-4$. Since $\mathbb{CH}^{n-2}$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{CH}^n$, there exists an orthogonal projection map $\pi : \mathbb{CH}^n \to \mathbb{CH}^{n-2}$. This map $\pi$ is a fiber bundle whose fibers are totally geodesic 4-planes isometric to $\mathbb{CH}^2$.

For $r > 0$ let $E(r)$ denote the $r$-neighborhood of $\mathbb{CH}^{n-2}$. Then $E(r)$ is a real hypersurface in $\mathbb{CH}^n$, and consequently we can decompose $\mathbf{c_n}$ as

$$\mathbf{c_n} = (\mathbf{c_n})_r + dr^2$$

where $(\mathbf{c_n})_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r : E(r) \to \mathbb{CH}^{n-2}$ denote the restriction of $\pi$ to $E(r)$. Note that $\pi_r$ is an $\mathbb{S}^3$-bundle whose fiber over any point $q \in \mathbb{CH}^{n-2}$ is (topologically) the 3-sphere of radius $r$ in the totally geodesic 4-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where

$\mathcal{V}(r)$ is tangent to the 3-sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$.

For $r, s > 0$ there exists a diffeomorphism $\phi_{sr} : E(s) \to E(r)$ induced by the geodesic flow along the totally geodesic 4-planes orthogonal to $\mathbb{CH}^{n-2}$. Fix $p \in E(r)$ arbitrary, let $q = \pi(p) \in \mathbb{CH}^{n-2}$, and let $\gamma$ be the unit speed geodesic such that $\gamma(0) = q$ and $\gamma(r) = p$. In what follows, all computations are considered in the tangent space $T_pE(r)$.

Note that $\mathcal{V}(r)$ is tangent to both $E(r)$ and the totally geodesic 4-plane $\pi^{-1}(q)$. Then since $\pi^{-1}(q)$ is preserved by the geodesic flow, we have that $d\phi_{sr}$ takes $\mathcal{V}(s)$ to $\mathcal{V}(r)$. Consider the complex geodesic $P = exp(span(\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}))$, where $J$ denotes the complex structure on $\mathbb{CH}^n$. P intersects $E(r)$ orthogonally, and $P \cap E(r)$ is isometric to a circle of radius $r$. Thus, since a complex geodesic has curvature $-4$, there exists a suitable identification $P \cong \mathbb{S}^1 \times (0, \infty)$ where the metric $\mathbf{c_5}$ restricted to $P$ can be written as

$$\frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2$$

where $d\theta^2$ denotes the round metric on the unit circle $\mathbb{S}^1$. Note that the presence of the "1/4" is to make the metric complete when extended to the core $\mathbb{CH}^2$.

Notice that $\frac{\partial}{\partial\theta}$ is a vector field on the three sphere $\mathbb{S}^3$ mentioned above. More generally, thinking of $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ with respect to the usual Hermitian metric, there is an obvious free action of the circle $\mathbb{S}^1$ on $\mathbb{S}^3$. The unit tangent vector field with respect to this action corresponds to the vector field $\frac{\partial}{\partial\theta}$ above. This action fibers $\mathbb{S}^3$ over the complex projective line $\mathbb{CP}^1$ (which, here, is the Hopf fibration), and the Riemannian submersion metric on this fiber bundle is an example of a Berger sphere. Let $\alpha(t)$ be a unit speed geodesic in $\mathbb{CP}^1$ such that $\alpha(0) = p$. Then $\exp_p(\alpha'(0), \frac{\partial}{\partial r})$ forms a totally real totally geodesic 2-plane in $\mathbb{CH}^n$. Thus the curvature of this 2-plane is $-1$. Since the direction of $\alpha$ was arbitrary, we can write the Riemannian metric $(\mathbf{c_n})_r$ restricted to $\mathcal{V}(r)$ as

$$\sinh^2(r)\mathbf{p_1} + \frac{1}{4}\sinh^2(2r)d\theta^2$$

where $\mathbf{p_1}$ denotes the complex projective metric on $\mathbb{CP}^1$.

Now let $\beta(t)$ be any unit speed geodesic in $\mathbb{CH}^{n-2}$ such that $\beta(0) = q$. Then $Q = exp(span(\beta'(0), \gamma'(0)))$ is a totally real totally geodesic submanifold of $\mathbb{CH}^n$, and thus $K(\beta', \gamma') = -1$. Therefore, the metric $\mathbf{c_n}$ restricted to $Q$ can be written as $\cosh^2(r)dt^2 + dr^2$. But since $\gamma$ was arbitrary, we can write the metric on the (2n-3)-dimensional submanifold determined by $\mathbb{CH}^{n-2}$ and $\frac{\partial}{\partial r}$ as $\cosh^2 \mathbf{c_{n-2}} + dr^2$. This leads to the following (compare to Theorem 2.1 of [Min18]).

**Theorem 2.1.** *The complex hyperbolic manifold $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2}$ can be written as $E \times (0, \infty)$, where $E \cong \mathbb{CH}^{n-2} \times \mathbb{S}^3$ equipped with the metric*

$$\mathbf{c_n} = \cosh^2(r)\mathbf{c_{n-2}} + \sinh^2(r)\mathbf{p_1} + \frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2. \tag{2.1}$$

2.2. **The warped product metric, orthonormal basis, and curvature formulas in $\mathbb{CH}^n \backslash \mathbb{CH}^{n-2}$.** For some positive, increasing real-valued functions $h, v, v_r : (0, \infty) \to \mathbb{R}$ define the Riemannian metric

$$\gamma := h^2(r)\mathbf{c_{n-2}} + v^2(r)\mathbf{p_1} + \frac{1}{4}v_r^2(r)d\theta^2 + dr^2. \tag{2.2}$$

Of course, this is just the warped product metric associated to equation (2.1), and so $\gamma = \mathbf{c_n}$ when $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$.

For the remainder of this Section, fix $p = (q_1, \bar{q}, r) \in \mathbb{CH}^{n-2} \times \mathbb{S}^3 \times (0, \infty) \cong \mathbb{CH}^n \setminus \mathbb{CH}^{n-2}$. Also, write $\bar{q} = (q_2, \theta)$ where $q_2 \in \mathbb{CP}^1$ and $\theta \in \mathbb{S}^1$ (thinking of $\mathbb{S}^3$ as the total space of the Hopf fibration). Let $(\check{X}_1, \check{X}_2, \dots, \check{X}_{2n-4})$ be an orthonormal collection of vector fields near $q_1 \in \mathbb{CH}^{n-2}$ which satisfies:

(1) $[\check{X}_i, \check{X}_j]_{q_1} = 0$ for all $1 \leq i, j \leq 2n-4$.
(2) $J\check{X}_{i+1}|_{q_1} = \check{X}_i|_{q_1}$ for all $2 \leq i \leq 2n-4$ and $i$ even.

Define an analogous collection of vector fields $(\check{X}_{2n-3}, \check{X}_{2n-2})$ about $q_2 \in \mathbb{CP}^1$ so that $J\check{X}_{2n-2}|_{q_2} = \check{X}_{2n-3}|_{q_2}$ and $[\check{X}_{2n-2}, \check{X}_{2n-3}]_{q_2} = 0$. Extend both collections to vector fields $(X_1, \dots, X_{2n-2})$ near $p$. Lastly, let $X_{2n-1} = \frac{\partial}{\partial \theta}$ and $X_{2n} = \frac{\partial}{\partial r}$.

Define an orthonormal basis $\{Y_i\}_{i=1}^{2n}$ with respect to $\gamma$ by

$$Y_i = \frac{1}{h} X_i \quad (\text{for } 1 \leq i \leq 2n-4) \qquad Y_j = \frac{1}{v} X_j \quad (\text{for } j = 2n-3, 2n-2) \tag{2.3}$$
$$Y_{2n-1} = \frac{1}{\frac{1}{2} v_r} X_{2n-1} \qquad Y_{2n} = X_{2n}.$$

We then have the following Theorem which states all nonzero terms of the $(4,0)$ curvature tensor $R_\gamma$ (up to the symmetries of the curvature tensor) with respect to the basis $(Y_i)$ (this is a special case of Theorem 2.3 of [Min18]).

**Theorem 2.2.** *In terms of the basis given in equation* (2.3)*, the only independent nonzero components of the* $(4,0)$ *curvature tensor* $R_\gamma$ *are given by the following formulas. In these formulas:* $i, i' \in \{1, \dots, 2n-4\}$ *with* $|JX_i| \neq |JX_{i'}|$, $k \in \{2n-3, 2n-2\}$, $(i,j) \neq (i',j') \in \{(1,2), \dots, (2n-5, 2n-4)\}$, $(k,l) = (2n-3, 2n-2)$, *and* $R^\gamma_{abcd} := \langle R_\gamma(Y_a, Y_b)Y_c, Y_d \rangle_\gamma$.

$$R^\gamma_{ijij} = -\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3v_r^2}{4h^4} \qquad R^\gamma_{klkl} = -\left(\frac{v'}{v}\right)^2 + \frac{4}{v^2} - \frac{3v_r^2}{4v^4}$$

$$R^\gamma_{ii'ii'} = -\left(\frac{h'}{h}\right)^2 - \frac{1}{h^2} \qquad R^\gamma_{ikik} = -\frac{h'v'}{hv}$$

$$R^\gamma_{i,2n-1,i,2n-1} = -\frac{h'v_r'}{hv_r} + \frac{v_r^2}{4h^4} \qquad R^\gamma_{k,2n-1,k,2n-1} = -\frac{v'v_r'}{vv_r} + \frac{v_r^2}{4v^4}$$

$$R^\gamma_{i,2n,i,2n} = -\frac{h''}{h} \qquad R^\gamma_{k,2n,k,2n} = -\frac{v''}{v} \qquad R^\gamma_{2n-1,2n,2n-1,2n} = -\frac{v_r''}{v_r}$$

$$R^\gamma_{iji'j'} = 2R^\gamma_{ii'jj'} = -2R^\gamma_{ij'ji'} = -\frac{2}{h^2} - \frac{v_r^2}{2h^4}$$

$$R^\gamma_{ijkl} = 2R^\gamma_{ikjl} = -2R^\gamma_{iljk} = -\frac{v_r^2}{2h^2v^2}$$

$$R^\gamma_{i,j,2n-1,2n} = 2R^\gamma_{i,2n-1,j,2n} = -2R^\gamma_{i,2n,j,2n-1} = -\frac{v_r}{h^2}\left(\ln \frac{v_r}{h}\right)'$$

$$R^\gamma_{k,l,2n-1,2n} = 2R^\gamma_{k,2n-1,l,2n} = -2R^\gamma_{k,2n,l,2n-1} = -\frac{v_r}{v^2}\left(\ln \frac{v_r}{v}\right)'.$$

## 3. Computing sectional curvatures of generic 2-planes

The negatively curved codimension one distribution $\mathscr{D}$ from Theorem 1.1 is the distribution orthogonal to $\frac{\partial}{\partial\theta}$ (or equivalently $Y_{2n-1}$). Let $p = (q_1, q_2, \theta, r)$ as in the previous Section. Note that $\mathscr{D}$ is not integrable, since $[X_{2n-3}, X_{2n-2}] = 2\frac{\partial}{\partial\theta}$ (see Lemma 4.2 of [Min18]). To prove that the sectional curvature of $\mathscr{D}$ is bounded above by a negative constant, it is enough to consider the "generic" case of 2-planes $\sigma \subset T_p\mathbb{CH}^n$ whose projection onto both $T_{q_1}\mathbb{CH}^{n-2}$ and $T_{q_2}\mathbb{CP}^1$ is at least 1-dimensional.

To simplify our curvature calculations, we choose the frame $(Y_i)$ depending on the position of $\sigma$. The homeomorphism $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2} \cong \mathbb{CH}^{n-2} \times \mathbb{S}^3 \times (0,\infty)$ gives a natural identification of $\mathscr{D}_p$ with $T_{q_1}\mathbb{CH}^{n-2} \times T_{q_2}\mathbb{CP}^1 \times \mathbb{R}$. Since the $T_{q_1}\mathbb{CH}^{n-2} \times T_{q_2}\mathbb{CP}^1$ component of $\mathscr{D}_p$ has codimension one, $\sigma$ contains a unit vector $A$ which is contained in this component. Define the projections $\pi_1 : \mathscr{D}_p \to T_{q_1}\mathbb{CH}^{n-2}$ and $\pi_2 : \mathscr{D}_p \to T_{q_2}\mathbb{CP}^1$. Let $\check{X}_1 \in T_{q_1}\mathbb{CH}^{n-2}$ and $\check{X}_5 \in T_{q_2}\mathbb{CP}^1$ be unit vectors parallel to $\pi_1(A)$ and $\pi_2(A)$, respectively. Next, define $\check{X}_2 \in T_{q_1}\mathbb{CH}^{n-2}$ so that $J\check{X}_2 = \check{X}_1$ and $\check{X}_6 \in T_{q_2}\mathbb{CP}^1$ so that $J\check{X}_6 = \check{X}_5$. Finally, if necessary, we choose $\check{X}_3 \in T_{q_1}\mathbb{CH}^{n-2}$ so that $(\check{X}_1, \check{X}_2, \check{X}_3)$ is an orthonormal collection of vectors whose span contains $\pi_1(\sigma)$.

As in Section 2, extend these vectors to vector fields near $q_1$ and $q_2$ with (by abusing notation) the same name, and then extend them to vector fields $X_i$ near $p$. Let $Y_i = (1/h)X_i$ for $i = 1, 2, 3$, $Y_j = (1/v)X_j$ for $j = 5, 6$, and $Y_8 = \frac{\partial}{\partial r}$ (where we omit $Y_7$ since $\mathscr{D}$ omits $\frac{\partial}{\partial\theta}$). By construction, $A$ can be written as a linear combination of $X_1$ and $X_5$. If $B \in \sigma$ is a unit vector orthogonal to $A$, then $(A, B)$ is an orthonormal basis of $\sigma$ which can be written as

$$A = a_1Y_1 + a_5Y_5 \qquad B = b_1Y_1 + b_2Y_2 + b_3Y_3 + b_5Y_5 + b_6Y_6 + b_8Y_8$$

where

$$a_1^2 + a_5^2 = 1 = b_1^2 + b_2^2 + b_3^2 + b_5^2 + b_6^2 + b_8^2$$

and

$$a_1b_1 + a_5b_5 = 0.$$

We then compute

$$\begin{aligned} K_\gamma(\sigma) &= \langle R^\gamma(A,B)A, B\rangle_\gamma \\ &= a_1^2b_2^2R^\gamma_{1212} + a_1^2b_3^2R^\gamma_{1313} + a_1^2b_6^2R^\gamma_{1616} + a_1^2b_8^2R^\gamma_{1818} + (a_1b_5 - a_5b_1)^2R^\gamma_{1515} \\ &\quad + a_5^2b_2^2R^\gamma_{2525} + a_5^2b_3^2R^\gamma_{3535} + a_5^2b_6^2R^\gamma_{5656} + a_5^2b_8^2R^\gamma_{5858} \\ &\quad + 2a_1a_5b_2b_6R^\gamma_{1256} - 2a_1a_5b_2b_6R^\gamma_{1625}. \end{aligned} \tag{3.1}$$

Note that by Theorem 2.2 we know that

$$2a_1a_5b_2b_6R^\gamma_{1256} - 2a_1a_5b_2b_6R^\gamma_{1625} = 3a_1a_5b_2b_6R^\gamma_{1256}. \tag{3.2}$$

We then combine equation (3.2) and Theorem 2.2 with equation (3.1) to obtain the following.

$$\begin{aligned} K_\gamma(\sigma) &= a_1^2 b_2^2 \left[ -\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3v_r^2}{4h^4} \right] + a_1^2 b_3^2 \left[ -\left(\frac{h'}{h}\right)^2 - \frac{1}{h^2} \right] + a_1^2 b_8^2 \left( -\frac{h''}{h} \right) \\ &+ \left[ a_1^2 b_6^2 + (a_1 b_5 - a_5 b_1)^2 + a_5^2 b_2^2 + a_5^2 b_3^2 \right] \left( -\frac{h'v'}{hv} \right) + a_5^2 b_8^2 \left( -\frac{v''}{v} \right) \\ &+ a_5^2 b_6^2 \left[ -\left(\frac{v'}{v}\right)^2 + \frac{4}{v^2} - \frac{3v_r^2}{4v^4} \right] + 3a_1 b_2 a_5 b_6 \left( \frac{-v_r^2}{2h^2 v^2} \right). \end{aligned} \tag{3.3}$$

Note that only the 6th and 7th terms (the last two terms) of equation (3.3) are capable of being positive. One method to ensure that the 6th term is negative is to choose $v$ sufficiently small (independent of $v_r$). One can then deal with the 7th (mixed) term by ensuring that the 1st and 6th terms are sufficiently negative, since these two terms (combined) contain the same coefficients as the mixed term. In particular, any contribution from coefficients other than $a_1, a_5, b_2$, and $b_6$ will only ever make the curvature more negative. So the "worst case" situation for us is when $b_1 = b_3 = b_5 = b_8 = 0$. In this case, equation (3.3) reduces to

$$\begin{aligned} K_\gamma(\sigma) &= a_1^2 b_2^2 \left[ -\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3v_r^2}{4h^4} \right] + (a_1^2 b_6^2 + a_5^2 b_2^2) \left( -\frac{h'v'}{hv} \right) \\ &+ a_5^2 b_6^2 \left[ -\left(\frac{v'}{v}\right)^2 + \frac{4}{v^2} - \frac{3v_r^2}{4v^4} \right] + 3a_1 b_2 a_5 b_6 \left( \frac{-v_r^2}{2h^2 v^2} \right). \end{aligned} \tag{3.4}$$

This is the equation that we will use in the next section when proving that $g$, restricted to $\mathscr{D}$, has negative sectional curvature.

## 4. Constructing the metric in Theorem 1.1

In this Section we develop warping functions for $h(r)$, $v(r)$, and $v_r(r)$, defined for $r \in (-\infty, \infty)$, so that the resulting metric $g$ (where $g := \gamma$ from Theorem 2.2) satisfies the conditions of Theorem 1.1. The domain of $(-\infty, \infty)$ turns each component of $N$ into a "cusp" of $M$, which is to ensure that $g$ is complete. An outline of how we define these functions is as follows.

We begin by letting $\varepsilon > 0$ be much smaller than the normal injectivity radius of $N$. We also let $\delta > 0$ be a small positive constant and $a < 0$ be a large (in absolute value) negative constant. On the region $(-\infty, a)$ we define

$$h = \delta e^r \qquad v = \frac{1}{3}\varepsilon e^r \qquad v_r = \varepsilon e^{\frac{r}{2}}. \tag{4.1}$$

On the region $\left(a, \frac{1}{3}\varepsilon\right)$ we slowly warp $h$ from $\delta e^r$ to $\cosh(r)$ while keeping $v = \frac{1}{3}\varepsilon e^r$ and $v_r = \varepsilon e^{\frac{r}{2}}$. We then "bend" $v$ to $\sinh(r)$ in a small neighborhood of $r = \frac{1}{3}\varepsilon$. These functions remain as they are on the interval $\left(\frac{1}{3}\varepsilon, \frac{1}{2}\varepsilon\right)$. In a small neighborhood of $r = \frac{1}{2}\varepsilon$ we then "bend" $v_r$ from $\varepsilon e^{\frac{r}{2}}$ to $\sinh(2r)$. Finally, for $r > \varepsilon$ we have $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$ so that $g$ agrees with the complex hyperbolic metric on this region.

A quick remark before we get into some more details is that the definitions for our warping functions for the interval $(-\infty, a)$ in equation (4.1) guarantee that $g$ still has finite volume.

In the following Subsections we will prove that $g$ has negative curvature in each of the intervals $(-\infty, a)$, $\left(a, \frac{1}{3}\varepsilon\right)$, and $\left(\frac{1}{3}\varepsilon, \frac{1}{2}\varepsilon\right)$ for $\varepsilon, \delta > 0$ chosen sufficiently small and $a < 0$ chosen sufficiently large negative. But let us now quickly discuss why we can disregard the endpoints of each interval.

Note that the functions $\frac{1}{3}\varepsilon e^r$ and $\sinh(r)$ intersect at approximately $r = \frac{1}{3}\varepsilon$, and $\varepsilon e^{\frac{1}{2}r} = \sinh(2r)$ at approximately $r = \frac{1}{2}\varepsilon$. So the use of the term "bend" in our discussion above really meant to apply the following Lemma from [Bel11] to the concatenation of the two functions.

**Lemma 4.1.** *Given real numbers $k, a_1, c, a_2$ with $a_1 < c < a_2$, let $f_1 : [a_1, c] \to \mathbb{R}$ and $f_2 : [c, a_2] \to \mathbb{R}$ be $C^2$ functions satisfying $f_i'' \geq k$, $f_1(c) = f_2(c)$, and $f_1'(c) \leq f_2'(c)$. If $f : [a_1, a_2] \to \mathbb{R}$ denotes the concatenation of $f_1$ and $f_2$, then for any small $\delta > 0$ there exists a $C^2$ function $f_\delta : [a_1, a_2] \to \mathbb{R}$ such that*

1. *$f_\delta'' > k$.*
2. *$f_\delta = f$ and $f_\delta' = f'$ at the points $a_1$ and $a_2$.*
3. *if $f$ is increasing, then $f_\delta' > 0$.*
4. *if $f$ is $C^l$ on $[a_1, a_2]$ for some integer $l \in [0, \infty]$, then $f_\delta$ if $C^l$ on $[a_1, a_2]$, and $f_\delta$ converges to $f$ in the $C^l$-topology on $[a_1, a_2]$ as $\delta \to 0$.*

Since the sectional curvature of $g$ will be bounded above by a negative constant on each interval, we can apply Lemma 4.1 to arbitrarily small neighborhoods of $r = \frac{1}{3}\varepsilon$ and $r = \frac{1}{2}\varepsilon$ and maintain the fact that $g$ has negative sectional curvature. We will define $h$ to be a smooth function near $r = a$, and so there will be no issues at that endpoint either.

### 4.1. The interval $(-\infty, a)$.

On this region we have

$$h = \delta e^r \qquad v = \frac{1}{3}\varepsilon e^r \qquad v_r = \varepsilon e^{\frac{r}{2}}.$$

Plugging these directly into equation (3.4), we obtain:

$$\begin{aligned} K_\gamma(\sigma) &= a_1^2 b_2^2 \left[-1 - \frac{4}{\delta^2 e^{2r}} - \frac{3\varepsilon^2}{4\delta^4 e^{3r}}\right] - (a_1^2 b_6^2 + a_5^2 b_2^2) \\ &\quad + a_5^2 b_6^2 \left[-1 + \frac{36}{\varepsilon^2 e^{2r}} - \frac{243}{4\varepsilon^2 e^{3r}}\right] - 3a_1 a_5 b_2 b_6 \left(\frac{9}{2\delta^2 e^{3r}}\right). \end{aligned} \tag{4.2}$$

Now, since $r \leq a << 0$, the terms containing an $e^{3r}$ in the denominator will dominate this sum. Isolating those terms, we have

$$\begin{aligned} K_\gamma(\sigma) &\approx -a_1^2 b_2^2 \left(\frac{3\varepsilon^2}{4\delta^4 e^{3r}}\right) - a_5^2 b_6^2 \left(\frac{243}{4\varepsilon^2 e^{3r}}\right) - a_1 a_5 b_2 b_6 \left(\frac{27}{2\delta^2 e^{3r}}\right) \\ &= \frac{1}{e^{3r}} \left[-a_1^2 b_2^2 \left(\frac{3\varepsilon^2}{4\delta^4}\right) - a_5^2 b_6^2 \left(\frac{243}{4\varepsilon^2}\right) - a_1 a_5 b_2 b_6 \left(\frac{27}{2\delta^2}\right)\right]. \end{aligned} \tag{4.3}$$

We claim that, for any fixed $\varepsilon > 0$, there exists $\delta > 0$, such that the term inside the brackets in (4.3) is nonpositive. To see this, first note that if either $a_1 = 0$ or $b_2 = 0$, then the third term (the mixed term) is 0 and thus this sum has to be nonpositive for any choice of $\delta$. So let us now fix $a_1, a_5, b_2$, and $b_6$ with both $a_1$ and $b_2$ not zero. Then since the first term is nonzero and has a $\delta^4$ in its denominator, we can choose $\delta$ sufficiently small to ensure that the entire term is negative. Since the possible selections for $a_1, a_5, b_2$, and $b_6$ is compact (recall $a_1^2 + a_5^2 = 1 = b_2^2 = b_6^2$), there exists a positive $\delta$ which works for all values of the parameters.

Let us remark that, even though (4.3) can be zero (which happens, for example, when $a_1 = 1 = b_6$ and $a_5 = b_2 = 0$), the sectional curvature tensor is still bounded above by a negative constant on this region. This is clear if we go back to the original curvature equation (4.2). If (4.3) is zero, then one of $a_1^2 b_6^2$ or $a_5^2 b_2^2$ is not zero. This ensures that (4.2) is still negative.

4.2. **The interval $\left(a, \frac{1}{3}\varepsilon\right)$.** Over this region we keep the functions

$$v = \frac{1}{3}\varepsilon e^r \qquad\qquad v_r = \varepsilon e^{\frac{r}{2}}$$

while slowly warping $h$ from $\delta e^r$ to $\cosh(r)$. More specifically, we define $h$ to be a smooth function on $\left(-\infty, \frac{1}{3}\varepsilon\right)$ which satisfies:

(1) $h = \delta e^r$ on $(-\infty, a)$.

(2) $h(r) \approx \delta e^a \left(1 - \dfrac{\alpha}{b-a}\right) + \cosh\left(\dfrac{1}{3}\varepsilon\right)\left(\dfrac{\alpha}{b-a}\right)$

for all $r \in \left(a, \dfrac{1}{3}\varepsilon\right)$ and where $\alpha = r - a \in \left(0, \dfrac{1}{3}\varepsilon - a\right)$.

(3) $h'(r) \approx \dfrac{\cosh\left(\frac{1}{3}\varepsilon\right) - \delta e^a}{\frac{1}{3}\varepsilon - a} < \sinh\left(\dfrac{1}{3}\varepsilon\right) \qquad$ for all $r \in \left(a, \dfrac{1}{3}\varepsilon\right)$.

(4) $h''(r) > 0$ small for all $r \in \left(a, \dfrac{1}{3}\varepsilon\right)$.

Such a function $h$ clearly exists. One should note that, for $\varepsilon > 0$ fixed, we choose $a$ sufficiently large (negative) to guarantee the inequality in (3).

Plugging these functions into equation (3.4), we obtain

$$K_\gamma(\sigma) = a_1^2 b_2^2 \left[-\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3\varepsilon^2 e^r}{4h^4}\right] + (a_1^2 b_6^2 + a_5^2 b_2^2)\left(-\frac{h'}{h}\right) \tag{4.4}$$

$$+ a_5^2 b_6^2 \left[-1 + \frac{36}{\varepsilon^2 e^{2r}} - \frac{243}{4\varepsilon^2 e^{3r}}\right] - \frac{27}{2} a_1 a_5 b_2 b_6 \left(\frac{1}{h^2 e^r}\right)$$

$$\approx a_1^2 b_2^2 \left[-\frac{4}{h^2} - \frac{3\varepsilon^2 e^r}{4h^4}\right] + a_5^2 b_6^2 \left[-1 + \frac{36}{\varepsilon^2 e^{2r}} - \frac{243}{4\varepsilon^2 e^{3r}}\right] \tag{4.5}$$

$$- \frac{27}{2} a_1 a_5 b_2 b_6 \left(\frac{1}{h^2 e^r}\right).$$

To show that (4.5) is always negative for $\varepsilon > 0$ chosen sufficiently small, we need to break up the interval $\left[a, \frac{1}{3}\varepsilon\right]$ at some large negative constant. Let us choose $-1000$. So we will first consider the interval $[a, -1000]$, and then the interval $\left[-1000, \frac{1}{3}\varepsilon\right]$.

So we first let $r \in [a, -1000]$, and consider the second summand of (4.5). Since $r$ is large negative and both variable terms have an "$\varepsilon^2$" in the denominator, the term with the $e^{3r}$ in the denominator dominates this second summand. Therefore this second summand is always negative, and becomes arbitrarily large negative as $\varepsilon$ approaches 0. We can then proceed as we did in the previous subsection. For any fixed $a_1, a_5, b_2$, and $b_6$, we can choose $\varepsilon > 0$ so that (4.5) is nonpositive. Then since the domain for those parameters is compact, we can choose $\varepsilon > 0$ so that (4.5) is nonpositive for all values of $a_1, a_5, b_2$, and $b_6$. Also, $K_\gamma(\sigma)$ is always bounded above by a negative constant due to the presence of the

$$(a_1^2 b_6^2 + a_5^2 b_2^2)\left(-\frac{h'}{h}\right)$$

term in (4.4).

Let us now consider the case when $r \in \left[-1000, \frac{1}{3}\varepsilon\right]$. The difference now is that the $e^{3r}$ term in the second summand does not necessarily dominate the $e^{2r}$ term. But this second summand is still always negative since $\frac{243}{4} > 36$ (and since $e^{2r} \approx e^{3r} \approx 1$ when $r \approx 0$). And moreover, the second summand of (4.5) still becomes arbitrarily large negative as $\varepsilon > 0$ is chosen sufficiently small. So for any fixed values for $a_1, a_5, b_2$, and $b_6$ and for a *fixed* choice for $r$, we can choose $\varepsilon > 0$ so that (4.5) is nonpositive. But since the domain for the parameters $a_1, a_5, b_2, b_6$, and $r$ is compact, we can choose $\varepsilon > 0$ so that (4.5) is nonpositive for all choices of $a_1, a_5, b_2$, and $b_6$ and for all $r \in \left[-1000, \frac{1}{3}\varepsilon\right]$. Of course, the interval $\left[-1000, \frac{1}{3}\varepsilon\right]$ depends on $\varepsilon$. But the interval decreases in size as $\varepsilon$ approaches zero, and so this does not effect the above compactness argument.

For a discussion on why we needed to break up the interval at $-1000$, please see Subsection 4.4 below.

Finally, in a small neighborhood of $r = \frac{1}{3}\varepsilon$, we use Lemma 4.1 to bend $v$ from $\frac{1}{3}\varepsilon e^r$ to $\sinh(r)$.

4.3. **The interval $\left(\frac{1}{3}\varepsilon, \frac{1}{2}\varepsilon\right)$.** Over this region we have

$$h = \cosh(r) \qquad v = \sinh(r) \qquad v_r = \varepsilon e^{\frac{r}{2}}.$$

When we evaluate (3.4) with these function values, we get

$$\begin{aligned} K_\gamma(\sigma) = {} & a_1^2 b_2^2 \left[-\frac{\sinh^2(r)}{\cosh^2(r)} - \frac{4}{\cosh^2(r)} - \frac{3\varepsilon^2 e^r}{4\cosh^4(r)}\right] - (a_1^2 b_6^2 + a_5^2 b_2^2) \\ & + a_5^2 b_6^2 \left[-\frac{\cosh^2(r)}{\sinh^2(r)} + \frac{4}{\sinh^2(r)} - \frac{3\varepsilon^2 e^r}{4\sinh^4(r)}\right] + \frac{3}{2} a_1 a_5 b_2 b_6 \left(\frac{-\varepsilon^2 e^r}{\sinh^2(r)\cosh^2(r)}\right). \end{aligned}$$

Since $r$ is near 0, we use the 1st order Taylor approximations

$$\cosh(r) \approx 1 \qquad \sinh(r) \approx r \qquad e^r \approx 1 + r$$

to obtain

$$\begin{aligned} K_\gamma(\sigma) \approx {} & a_1^2 b_2^2 \left[-r^2 - 4 - \frac{3}{4}\varepsilon^2(1+r)\right] - (a_1^2 b_6^2 + a_5^2 b_2^2) \\ & + a_5^2 b_6^2 \left[-\frac{1}{r^2} + \frac{4}{r^2} - \frac{3\varepsilon^2(1+r)}{4r^4}\right] + \frac{3}{2} a_1 a_5 b_2 b_6 \left(\frac{-\varepsilon^2(1+r)}{r^2}\right) \\ \approx {} & a_1^2 b_2^2 \left[-4 - \frac{3}{4}\varepsilon^2(1+r)\right] - (a_1^2 b_6^2 + a_5^2 b_2^2) \\ & + a_5^2 b_6^2 \left[\frac{3}{r^2} - \frac{3\varepsilon^2}{4r^3} - \frac{3\varepsilon^2}{4r^4}\right] + \frac{3}{2} a_1 a_5 b_2 b_6 \left(-\frac{\varepsilon^2}{r} - \frac{\varepsilon^2}{r^2}\right). \end{aligned} \tag{4.6}$$

We can write $r = k\varepsilon$ for some $k \in \left[\frac{1}{3}, \frac{1}{2}\right]$. Substituting this into (4.6) yields

$$\begin{aligned} K_\gamma(\sigma) &\approx a_1^2 b_2^2 \left[-4 - \frac{3}{4}\varepsilon^2(1 + k\varepsilon)\right] - (a_1^2 b_6^2 + a_5^2 b_2^2) \\ &\quad + a_5^2 b_6^2 \left[\frac{3}{k^2\varepsilon^2} - \frac{3}{4k^3\varepsilon} - \frac{3}{4k^4\varepsilon^2}\right] - \frac{3}{2} a_1 a_5 b_2 b_6 \left(\frac{1}{k^2} + \frac{\varepsilon}{k^2}\right) \\ &\approx -4a_1^2 b_2^2 - (a_1^2 b_6^2 + a_5^2 b_2^2) + a_5^2 b_6^2 \left[\frac{-3 + 12k^2}{4k^4\varepsilon^2} - \frac{3}{4k^3\varepsilon}\right] \qquad (4.7) \\ &\quad - \frac{3}{2k^2} a_1 a_5 b_2 b_6. \end{aligned}$$

Consider the third term of equation (4.7). Since we will choose $\varepsilon$ small, the term with the $\varepsilon^2$ in the denominator (generally) dominates the third summand. But notice that the numerator is always $\leq 0$ for $k \in \left[\frac{1}{3}, \frac{1}{2}\right]$, and is only zero for $k = \frac{1}{2}$. For $k$ arbitrarily close to $\frac{1}{2}$, the second term becomes the larger term and it becomes arbitrarily large negative as $\varepsilon \to 0$. The point here is that the quantity

$$\frac{-3 + 12k^2}{4k^4\varepsilon^2} - \frac{3}{4k^3\varepsilon} \qquad \text{for } k \in \left[\frac{1}{3}, \frac{1}{2}\right]$$

approaches $-\infty$ as $\varepsilon \to 0$. We can therefore apply a similar argument to what we have previously. Since the domain for the parameters $a_1, a_5, b_2$, and $b_6$ is compact, we can choose $\varepsilon > 0$ sufficiently small so that (4.7) is bounded above by a negative constant.

4.4. **Order in which we choose the parameters $\delta$, $a$, and $\varepsilon$.** The purpose of this Subsection is to ensure that we can choose the parameters $\delta$, $a$, and $\varepsilon$ as required above.

We first start with $a < -1000$ not yet fixed. We then fix $\varepsilon > 0$ so that $K_\gamma(\sigma)$ is bounded above by a negative constant on the intervals $[a, -1000]$, $\left[-1000, \frac{1}{3}\varepsilon\right]$, and $\left[\frac{1}{3}\varepsilon, \frac{1}{2}\varepsilon\right]$, and recall that the selection of $\varepsilon$ was independent of $r$ in the interval $[a, -1000]$. Similarly we choose $\delta > 0$ so that $K_\gamma(\sigma)$ is bounded above by a negative constant on the interval $(-\infty, a)$, and recall that this selection of $\delta$ was also independent of $r$.

We then (if necessary) choose $a$ larger (negative) so that condition (3) in the definition of $h(r)$ in the interval $\left[a, \frac{1}{3}\varepsilon\right]$ is satisfied. That is, so that

$$h'(r) \approx \frac{\cosh\left(\frac{1}{3}\varepsilon\right) - \delta e^a}{\frac{1}{3}\varepsilon - a} < \sinh\left(\frac{1}{3}\varepsilon\right) \text{ for all } r \in \left(a, \frac{1}{3}\varepsilon\right).$$

This condition is necessary in order to apply Lemma 4.1 at the endpoint $r = \frac{1}{3}\varepsilon$. So it is necessary that we choose $a$ after fixing $\varepsilon$.

The reason that we needed to break the interval $\left[a, \frac{1}{3}\varepsilon\right]$ at a large negative number is because we needed to use a compactness argument for $r$ on the region containing zero. But we couldn't use this on the entire interval, since choosing $a$ after fixing $\varepsilon$ expands the interval.

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