COMPLEX HYPERBOLIC GROMOV-THURSTON MANIFOLDS

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ABSTRACT. This is a preliminary version.

1. Introduction

Theorem 1.1. *Suppose that M is a Riemannian manifold modeled on $\mathbb{CH}^n$, and that N is a codimension two totally geodesic submanifold of M modeled on $\mathbb{CH}^{n-1}$. Furthermore, assume that the d-fold cyclic ramified cover X of M about N is a smooth manifold. Then X admits a smooth Riemannian metric whose sectional curvature is bounded above by a negative constant, provided that the normal injectivity radius of N in M is sufficiently large.*

2. Hyperbolic and complex hyperbolic metrics in polar coordinates

In this Section we describe the curvature formulas for $\mathbb{H}^n$ and $\mathbb{CH}^n$ written in terms of polar coordinates about a copy of $\mathbb{H}^{n-2}$ or $\mathbb{CH}^{n-1}$, respectively. The $(\mathbb{H}^n, \mathbb{H}^{n-2})$ case is from [Bel12], but can also be found in [LMMT] and [Min17b]. The curvature formulas for $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$ were derived in [Bel11] for curvatures in $[-1, -1/4]$, and converted to curvatures in $[-4, -1]$ in [Min17b]. In what follows we will simply stick with the model cases $(\mathbb{H}^n, \mathbb{H}^{n-2})$, and $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$, but of course all of these computations will "descend" to the pair (M, N).

2.1. The metric on $\mathbb{H}^n$ in polar coordinates about $\mathbb{H}^{n-2}$.

Let $\mathbb{H}^{n-2}$ denote a totally geodesic codimension two submanifold in $\mathbb{H}^n$, let r denote the distance to $\mathbb{H}^{n-2}$ within $\mathbb{H}^n$, and let $\mathbf{h_n}$, $\mathbf{h_{n-2}}$ denote the metrics on $\mathbb{H}^n$ and $\mathbb{H}^{n-2}$, respectively. Let $\phi_h : \mathbb{H}^n \to \mathbb{H}^{n-2}$ denote the orthogonal projection onto $\mathbb{H}^{n-2}$, and let $E_h(r)$ denote the r-tube about $\mathbb{H}^{n-2}$. It is well known (see for example [Bel12]) that with a suitable identification $E_h(r) \cong \mathbb{S}^1 \times \mathbb{R}^{n-2}$. The metric for $\mathbb{H}^n$ in polar coordinates about $\mathbb{H}^{n-2}$ is given by

$$\mathbf{h_n} = \cosh^2(r)\mathbf{h_{n-2}} + \sinh^2(r)d\theta^2 + dr^2 \tag{2.1}$$

where $d\theta^2$ denotes the round metric on the unit circle $\mathbb{S}^1$. Note that the metric in equation (2.1) is defined on $E \times [0, \infty)$, where E is an arbitrary r-tube as defined above.

A key feature of the real hyperbolic metric is the following. Let $p \in \mathbb{H}^{n-2}$ and let $\check{X}_1, \check{X}_2, \dots, \check{X}_{n-2}$ be a local orthonormal frame near p in $\mathbb{H}^{n-2}$ satisfying that $[\check{X}_i, \check{X}_j]_p = 0$ for all i, j. Let $q \in \mathbb{H}^n$ be such that $\phi_h(q) = p$. Extend the collection $(\check{X}_i)_{i=1}^{n-2}$ to vector fields $X_1, X_2, \dots, X_{n-2}$ defined near q in $\mathbb{H}^n$ via $d\phi_h^{-1}$, which are

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orthogonal to both $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$. Then the key property is that $[X_i, X_j]_q = 0$ for all i, j, or equivalently that the distribution determined by $(X_i)_{i=1}^{n-2}$ is integrable.

Let $v(r)$ and $h(r)$ be positive real-valued functions of r. Define the metric $\lambda := \lambda_{v,h}$ on $E \times (0,\infty)$ by

$$\lambda = h^2\mathbf{h_{n-2}} + v^2 d\theta^2 + dr^2.$$

Of course, when $v = \sinh(r)$ and $h = \cosh(r)$ we recover the hyperbolic metric $\mathbf{h_n}$. Fix vector fields $(X_i)_{i=1}^{n-2}$ as above. Let $X_{n-1} = \frac{\partial}{\partial\theta}$ and $X_n = \frac{\partial}{\partial r}$. Define the following orthonormal frame for λ:

$$Y_i = \frac{1}{h}X_i \;\text{ for } 1 \leq i \leq n-2 \qquad Y_{n-1} = \frac{1}{v}X_{n-1} \qquad Y_n = X_n.$$

Formulas for the components of the $(4,0)$ curvature tensor R_λ for λ are given by the following Theorem.

Theorem 2.1 (c.f. Section 2 of [Bel12])**.** *Let $R^\lambda_{i,j,k,l} := \langle R_\lambda(Y_i, Y_j)Y_k, Y_l\rangle_\lambda$. Then up to the symmetries of the curvature tensor, the only nonzero components of the (4,0) curvature tensor R_λ are the following:*

$$R^\lambda_{i,j,i,j} = -\frac{1}{h^2} - \left(\frac{h'}{h}\right)^2 \qquad R^\lambda_{i,n-1,i,n-1} = -\frac{h'v'}{hv}$$

$$R^\lambda_{i,n,i,n} = -\frac{h''}{h} \qquad R^\lambda_{n-1,n,n-1,n} = -\frac{v''}{v}$$

where $1 \leq i, j \leq n-2$.

One easily checks that plugging in the values $v(r) = \sinh(r)$ and $h(r) = \cosh(r)$ gives all sectional curvatures of -1.

2.2. **The metric on $\mathbb{CH}^n$ in polar coordinates about $\mathbb{CH}^{n-1}$.** Let $\mathbb{CH}^{n-1}$ denote a (real codimension two) complex line in $\mathbb{CH}^n$, let r denote the distance to $\mathbb{CH}^{n-1}$ within $\mathbb{CH}^n$, and let $\mathbf{c_n}$ and $\mathbf{c_{n-1}}$ denote the metrics on $\mathbb{CH}^n$ and $\mathbb{CH}^{n-1}$ normalized to have constant holomorphic curvature -4. Let $\phi_c : \mathbb{CH}^n \to \mathbb{CH}^{n-1}$ denote the orthogonal projection onto $\mathbb{CH}^{n-1}$, and let $E_c(r)$ denote the r-tube about $\mathbb{CH}^{n-1}$. It is proved in [Bel11] that with a suitable identification we have $E_c(r) \cong \mathbb{S}^1 \times \mathbb{R}^{2n-2}$. The metric in $\mathbb{CH}^n$ in polar coordinates about $\mathbb{CH}^{n-1}$ is then given by

$$\mathbf{c_n} = \cosh^2(r)\mathbf{c_{n-1}} + \frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2. \tag{2.2}$$

Note that the presence of the $1/4$ in the $d\theta^2$ term is so that $\mathbf{c_n}$ is complete, or equivalently so that $\mathbf{c_n}$ has total angle of 2π about the core copy of $\mathbb{CH}^{n-1}$.

Just as above, we need to understand the Lie brackets associated to the metric in (2.2). Let $p \in \mathbb{CH}^{n-1}$ and let $\check{X}_1, \check{X}_2, \ldots, \check{X}_{2n-2}$ be a local orthonormal frame near p in $\mathbb{CH}^{n-1}$ satisfying that $[\check{X}_i, \check{X}_j]_p = 0$ for all i, j. Let $q \in \mathbb{CH}^n$ be such that $\phi_c(q) = p$. Extend the collection $(\check{X}_i)_{i=1}^{2n-2}$ to vector fields $X_1, X_2, \ldots, X_{2n-2}$ defined near q in $\mathbb{CH}^n$ via $d\phi_c^{-1}$, which are orthogonal to both $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$. Then for all $1 \leq i, j, \leq 2n-2$, there exist *structure constants* c_{ij} such that

$$[X_i, X_j] = c_{ij}\frac{\partial}{\partial\theta}. \tag{2.3}$$

Combining the work in Section 5 of [Bel11] with Lemma 4.2 in [Min17b], it is not hard to see that

$$-2 \leq c_{ij} \leq 2 \text{ for all } i \text{ and } j. \tag{2.4}$$

Also, $c_{ij} = 0$ if and only if (X_i, X_j) spans a totally real totally geodesic 2-plane, whereas $c_{ij} = \pm 2$ if and only if (X_i, X_j) spans a complex line.

Let $v(r)$ and $h(r)$ be positive real-valued functions of r. Define the metric $\mu := \mu_{v,h}$ on $E \times (0, \infty)$ by

$$\mu = h^2 \mathbf{c_{n-1}} + \frac{1}{4} v^2 d\theta^2 + dr^2. \tag{2.5}$$

Of course, when $v = \sinh(2r)$ and $h = \cosh(r)$ we recover the complex hyperbolic metric $\mathbf{c_n}$. Fix vector fields $(X_i)_{i=1}^{2n-2}$ as above. Let $X_{2n-1} = \frac{\partial}{\partial \theta}$ and $X_{2n} = \frac{\partial}{\partial r}$. Define the following orthonormal frame for μ:

$$Y_i = \frac{1}{h} X_i \text{ for } 1 \leq i \leq 2n-2 \qquad Y_{2n-1} = \frac{1}{\frac{1}{2}v} X_{2n-1} \qquad Y_{2n} = X_{2n}. \tag{2.6}$$

Formulas for the components of the $(4, 0)$ curvature tensor R_μ of μ are given by the following Theorem.

Theorem 2.2 (compare Sections 7 and 8 of [Bel11] with Theorem 4.3 of [Min17b])**.** *Let $R^{\mu}_{i,j,k,l} := \langle R_\mu(Y_i, Y_j)Y_k, Y_l\rangle_\mu$. We then have the following formulas for components of the (4,0) curvature tensor R_μ:*

$$R^{\mu}_{i,j,i,j} = -\frac{1}{h^2} - \left(\frac{h'}{h}\right)^2 - \frac{3c_{ij}^2}{4h^2} - \frac{3c_{ij}^2 v^2}{16h^4} \qquad R^{\mu}_{i,2n-1,i,2n-1} = -\frac{h'v'}{hv} + \frac{v^2}{4h^4}$$

$$R^{\mu}_{i,2n,i,2n} = -\frac{h''}{h} \qquad R^{\mu}_{2n-1,2n,2n-1,2n} = -\frac{v''}{v}$$

$$R^{\mu}_{i,j,2n-1,2n} = 2R^{\mu}_{i,2n-1,j,2n} = -2R^{\mu}_{i,2n,j,2n-1} = -c_{ij}\frac{v}{2h^2}\left(\ln \frac{v}{h}\right)'.$$

where $1 \leq i, j \leq 2n-2$.

It is a nice exercise to check that plugging in the values $v(r) = \sinh(2r)$, $h(r) = \cosh(r)$, and $c_{ij} = 0$ or ± 2 yields the correct values for the sectional curvature tensor (which one can compute via equation (5.1) below). Analogously, one can check that inserting the appropriate values for c_{ij} gives the corresponding formulas from [Min17b]. Lastly, the above formulas can be obtained from the formulas in Sections 7 and 8 of [Bel11] by making the substitutions $h \to \frac{1}{2}h$, $v \to \frac{1}{2}v$, and $c_{ij} \to \frac{c_{ij}}{4}$.

There is one more nonzero component of R_μ (up to the symmetries of the curvature tensor) which is not listed in Theorem 2.2: the mixed terms which come entirely from the core $\mathbb{CH}^{n-2}$. In what follows we will only need a specific form of this mixed term, which we describe now. Suppose that (X_i, X_j) and (X_k, X_l) span an orthogonal pair of complex lines. The orthogonality implies that all other pairings span totally real totally geodesic 2-planes. Then using the notation of Theorem 2.2, we have (c.f. Theorem 4.3 of [Min17b]):

$$R^{\mu}_{i,j,k,l} = 2R^{\mu}_{i,k,j,l} = -2R^{\mu}_{i,l,j,k} = \pm\left(\frac{2}{h^2} + \frac{v^2}{2h^4}\right). \tag{2.7}$$

2.3. **Curvature formulas in $\mathbb{CH}^n \setminus \mathbb{CH}^{n-1}$ with respect to a holomorphic basis.** In some of the metrics in Section 3, we will make the assumption that all of the structure constants in (2.3) are identically zero. Computing the components of the sectional curvature tensor is more complicated than just plugging $c_{ij} = 0$ into the equations in Theorem 2.2. The reason for this is because those curvature formulas were computed under a fixed sum of squares for the structure constants (see eqn 5.11 of [Bel11]). So in this Subsection we quickly reproduce the calculations of Sections 6 and 7 of [Bel11] for this specific situation.

For simplicity, we will restrict ourselves to a holomorphic basis of $\mathbb{CH}^{n-1}$. A *holomorphic basis* for $\mathbb{CH}^{n-1}$ is an orthonormal basis $(X_1, X_2, \dots, X_{2n-2})$ such that

(1) for every odd integer i, the pair (X_i, X_{i+1}) spans a complex line. Moreover, $JX_i = X_{i+1}$, where J denotes the complex structure on $\mathbb{CH}^n$.
(2) for j an odd integer different from i, the pair (X_i, X_j) spans a totally real totally geodesic 2-plane.

Now let (X_i), (Y_i), and μ be exactly as in equations (2.5) and (2.6), with the additional assumption that $(X_i)_{i=1}^{2n-2}$ is a holomorphic basis for the horizontal component of $T_q\mathbb{CH}^n$. For i an odd integer, define

$$[X_i, X_{i+1}] = c_i \frac{\partial}{\partial \theta}$$

and assume $[X_i, X_j] = 0$ if $j \neq i+1$. Note that this is just a special case of (2.3), since $c_{ij} = 0$ if (X_i, X_j) spans a totally real totally geodesic 2-plane.

The goal is to compute the components of the $(4,0)$ curvature tensor R_μ of μ with respect to this set-up. These equations are stated in the following Theorem.

Theorem 2.3. *Let $R^\mu_{i,j,k,l} := \langle R_\mu(Y_i, Y_j)Y_k, Y_l\rangle_\mu$, and let $(X_i)_{i=1}^{2n-2}$ be a holomorphic basis. We then have the following formulas for components of the (4,0) curvature tensor R_μ with respect to the corresponding orthonormal basis $(Y_i)_{i=1}^{2n}$:*

$$R^\mu_{i,j,i,j} = -\frac{1}{h^2} - \left(\frac{h'}{h}\right)^2 \qquad R^\mu_{i,2n-1,i,2n-1} = -\frac{h'v'}{hv} + \frac{c_i^2 v^2}{16h^4}$$

$$R^\mu_{i,i+1,i,i+1} = -\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3c_i^2 v^2}{16h^4}$$

$$R^\mu_{i,2n,i,2n} = -\frac{h''}{h} \qquad R^\mu_{2n-1,2n,2n-1,2n} = -\frac{v''}{v}$$

$$R^\mu_{i,i+1,2n-1,2n} = 2R^\mu_{i,2n-1,i+1,2n} = -2R^\mu_{i,2n,i+1,2n-1} = -c_i\frac{v}{2h^2}\left(\ln\frac{v}{h}\right)'$$

$$R^\mu_{i,i+1,k,k+1} = 2R^\mu_{i,k,i+1,k+1} = -2R^\mu_{i,k+1,i+1,k} = -\frac{2}{h^2} - \frac{c_i c_k v^2}{8h^4}$$

where $1 \leq i,j,k \leq 2n-2$, k is an odd integer different from i, and $j \neq i, i+1$. Also, any equations using both i and $i+1$ assume that i is an odd integer.

Proof. The proof here is analogous to Section 6 of [Bel11]. We compute the A and T tensors of the metric

$$\mu_r = h^2\mathbf{c_{n-1}} + \frac{1}{4}v^2 d\theta^2.$$

We then use Theorem 9.28 of [Bes87] to compute formulas for the components of the sectional curvature tensor R_{μ_r} of μ_r. Finally, we use equations (5.2) through (5.5) to obtain the components of R_μ.

First, by identical reasoning to [Bel11], the fibers of the Riemannian submersion are totally geodesic. Thus, the T tensor is identically zero. We now compute the A tensor. For this, we fix the notation that i is an odd integer and $j \neq i, i+1$.

By ([Bes87] eqn. 9.24) we have that

$$A_{X_i}X_{i+1} = \frac{1}{2}\mathscr{V}[X_i, X_{i+1}] = \frac{c_i}{2}X_{2n-1} \qquad \text{and} \qquad A_{X_i}X_j = 0.$$

Then by ([Bes87] eqn. 9.21d), we have

$$\begin{aligned}\langle A_{X_i}X_{2n-1}, X_{i+1}\rangle_{\mu_r} &= -\langle A_{X_i}X_{i+1}, X_{2n-1}\rangle_{\mu_r}\\ &= -\frac{c_i}{2}\langle X_{2n-1}, X_{2n-1}\rangle_{\mu_r} = -\frac{c_i}{8}v^2.\end{aligned}$$

In a similar manner we see that all other components of $A_{X_i}X_{2n-1}$ are zero. Thus, we have that

$$A_{X_i}X_{2n-1} = -\frac{c_i v^2}{8h^2}X_{i+1} \qquad \text{and} \qquad A_{X_{i+1}}X_{2n-1} = \frac{c_i v^2}{8h^2}X_i.$$

We are now ready to compute the components of R_{μ_r} in terms of the basis $(Y_i)_{i=1}^{2n-1}$. In the following calculations we use Theorem 9.28 from [Bes87], as well as the linearity of the curvature tensor. Also, in the fourth bullet point, j denotes an odd integer different from i.

- $\langle R_{\mu_r}(X_i, X_{2n-1})X_i, X_{2n-1}\rangle_{\mu_r} = \langle A_{X_i}X_{2n-1}, A_{X_i}X_{2n-1}\rangle_{\mu_r} = \dfrac{c_i^2 v^4}{64h^4}\cdot h^2 = \dfrac{c_i^2 v^4}{64h^2}$
  $$\implies \langle R_{\mu_r}(Y_i, Y_{2n-1})Y_i, Y_{2n-1}\rangle_{\mu_r} = \frac{4}{h^2v^2}\cdot\frac{c_i^2v^4}{64h^2} = \frac{c_i^2v^2}{16h^4}.$$
- $\langle R_{\mu_r}(X_i, X_{i+1})X_i, X_{i+1}\rangle_{\mu_r} = \langle \check{R}_{\mu_r}(\check{X}_i, \check{X}_{i+1})\check{X}_i, \check{X}_{i+1}\rangle_{\mu_r} - 3\langle A_{X_i}X_{i+1}, A_{X_i}X_{i+1}\rangle_{\mu_r}$
  $$= -4h^2 - \frac{3c_i^2v^2}{16}$$
  $$\implies \langle R_{\mu_r}(Y_i, Y_{i+1})Y_i, Y_{i+1}\rangle_{\mu_r} = \frac{1}{h^4}\cdot\left(-4h^2 - \frac{3c_i^2v^2}{16}\right) = -\frac{4}{h^2} - \frac{3c_i^2v^2}{16h^4}.$$
- $\langle R_{\mu_r}(X_i, X_j)X_i, X_j\rangle_{\mu_r} = \langle \check{R}_{\mu_r}(\check{X}_i, \check{X}_j)\check{X}_i, \check{X}_j\rangle_{\mu_r} - 3\langle A_{X_i}X_j, A_{X_i}X_j\rangle_{\mu_r} = -h^2$
  $$\implies \langle R_{\mu_r}(Y_i, Y_j)Y_i, Y_j\rangle_{\mu_r} = \frac{1}{h^4}\cdot(-h^2) = -\frac{1}{h^2}.$$
- $\langle R_{\mu_r}(X_i, X_{i+1})X_j, X_{j+1}\rangle_{\mu_r} = \langle \check{R}_{\mu_r}(\check{X}_i, \check{X}_{i+1})\check{X}_j, \check{X}_{j+1}\rangle_{\mu_r} - 2\langle A_{X_i}X_{i+1}, A_{X_j}X_{j+1}\rangle_{\mu_r}$
  $$= -2h^2 - 2\left(\frac{c_ic_j}{4}\cdot\frac{v^2}{4}\right) = -2h^2 - \frac{c_ic_jv^2}{8}$$
  $$\implies \langle R_{\mu_r}(Y_i, Y_{i+1})Y_j, Y_{j+1}\rangle_{\mu_r} = \frac{1}{h^4}\cdot\left(-2h^2 - \frac{c_ic_jv^2}{8}\right) = -\frac{2}{h^2} - \frac{c_ic_jv^2}{8h^4}.$$

A few quick remarks about the above calculations. All of the curvatures with a "hat" above the symbols occur in the horizontal fiber, which is isometric to the h^2-multiple of $\mathbf{c_{n-1}}$. One can use equation (5.1) below to verify that the corresponding curvatures above in $\mathbf{c_{n-1}}$ are indeed -4, -1, and -2. One can also deduce from Theorem 9.28 in [Bes87] that $R^{\mu_r}_{i+1,2n-1,i+1,2n-1} = R^{\mu_r}_{i,2n-1,i,2n-1}$, and that $R^{\mu_r}_{i,i+1,j,j+1} = 2R^{\mu_r}_{i,j,i+1,j+1} = -2R^{\mu_r}_{i,j+1,i_1,j}$ (and recall that this subscript notation is just the components of the curvature tensor with respect to the ON basis (Y_i)).

Combining the above computations with equations (5.2) through (5.5) below proves the Theorem.

$\square$

2.4. **Sectional Curvatures in $\mathbb{CH}^n \setminus \mathbb{CH}^{n-1}$.** The fact that R_μ has nonzero mixed terms adds difficulty to showing that the sectional curvature of μ is bounded above by a negative constant. The purpose of this Subsection is to provide two ways of proving this.

The first method, which will be used in tandem with Theorem 2.3, is the following Lemma. This Lemma is virtually identical to Lemma 9.1 of [Min16]. Its proof *is* identical, and so we omit it.

Lemma 2.4. *Using the notation of Theorem 2.3, suppose that*

$$R_{i,j,i,j} \leq -|R_{i,j,k,l}| \qquad \text{and} \qquad R_{k,l,k,l} \leq -|R_{i,j,k,l}|$$

for all $1 \leq i,j,k,l \leq 2n$. Furthermore, assume that if any of the above mixed terms are zero, then the corresponding inequalities are strict. Then there exists $C < 0$ such that $K(\sigma) < C$ for any 2-plane σ.

The second method is that described by Belegradek in Section 9 of [Bel11]. This will be used in conjunction with Theorem 2.2 to prove Theorem 3.2 below. We quickly describe this method now.

Let σ be a 2-plane in $T_q\mathbb{CH}^n$, and assume that $d\phi_c(\sigma)$ is a 2-plane in $T_p\mathbb{CH}^{n-2}$ (where $p = \phi_c(q)$). Note that this is a generic assumption: almost all 2-planes in $T_q\mathbb{CH}^n$ satisfy this assumption. So if we show that $K(\sigma)$ is bounded above by a negative constant, then all sectional curvatures are bounded above by a negative constant by continuity.

Following Section 9 of [Bel11], we can find orthonormal vector fields (X_1, X_2) near p and an orthonormal basis (A, B) of σ such that

$$A = a_1Y_1 + a_2Y_2 + a_3Y_{2n-1} + a_4Y_{2n} \qquad \text{and} \qquad B = b_1Y_1 + b_3Y_{2n-1}. \tag{2.8}$$

Then a direct computation shows that (c.f. equation (9.1) of [Bel11], where we substitute A and B for C and D)

$$\begin{aligned} K(\sigma) =&(a_1^2 + a_3^2)R^\mu_{1,2n-1,1,2n-1} + a_2^2b_3^2R^\mu_{2,2n-1,2,2n-1} \\ &+ a_4^2b_3^2R^\mu_{2n-1,2n,2n-1,2n} + a_4^2b_1^2R^\mu_{1,2n,1,2n} \\ &+ a_2^2b_1^2R^\mu_{1,2,1,2} + 3a_2a_4b_1b_3R^\mu_{2n,2n-1,1,2}. \end{aligned} \tag{2.9}$$

3. Four special metrics

All metrics in this Section are defined on the product $E \times [0,\infty) \cong \mathbb{R}^{2n-2} \times \mathbb{S}^1 \times [0,\infty)$ except for the last metric g_k, which is only defined on $E \times (0,\infty)$.

3.1. **The hyperbolized complex-hyperbolic metric g_{hch}.** The *hyperbolized complex-hyperbolic metric* $\mathbf{g_{hch}}$ is defined by setting $h(r) = \cosh(r)$ and $v(r) =$

$2\sinh(r)$ in equation (2.5), and keeping the same structure constants as the complex hyperbolic metric (that is, those defined in (2.3) and (2.4)). So

$$\mathbf{g_{hch}} = \cosh^2(r)\mathbf{c_{n-1}} + \sinh^2(r)d\theta^2 + dr^2.$$

The name of this metric comes from the fact that it has the same structure constants as the complex-hyperbolic metric, but has the same coefficients as the hyperbolic metric.

Note that, in the complex-hyperbolic metric, $v(r) = \sinh(2r) = 2\sinh(r)\cosh(r)$. Of course, $\cosh(0) = 1$, and so $\mathbf{g_{hch}}$ is a complete metric on $\mathbb{R}^{2n}$ which induces the complex-hyperbolic metric $\mathbf{c_{n-1}}$ on the core codimension two copy of $\mathbb{R}^{n-2}$. This fact is crucial for the warping construction in the following Section.

We now turn our attention to the sectional curvature tensor R_{hch} of $\mathbf{g_{hch}}$. The following Lemma just uses Theorem 2.2 to compute certain components of R_{hch}.

Lemma 3.1. *Using the notation of Theorem 2.2, we have:*

$$R^{hch}_{i,j,i,j} = -1 - \frac{3c_{ij}^2}{4\cosh^2(r)} - \frac{3c_{ij}^2\sinh^2(r)}{4\cosh^4(r)} \qquad R^{hch}_{i,2n-1,i,2n-1} = -1 + \frac{\sinh^2(r)}{\cosh^4(r)}$$

$$R^{hch}_{i,2n,i,2n} = -1 \qquad R^{hch}_{2n-1,2n,2n-1,2n} = -1 \qquad R^{hch}_{i,j,2n-1,2n} = -\frac{c_{ij}}{\cosh^3(r)}$$

where $1 \leq i,j \leq 2n-2$.

Note that $\mathbf{g_{hch}}$ is asymptotically hyperbolic as each term above containing a hyperbolic trigonometric function approaches zero as r approaches infinity (also, the term from (2.7) is equal to $\pm\left(\frac{2}{\cosh^2(r)} + \frac{2\sinh^2(r)}{\cosh^4(r)}\right)$ and similarly approaches zero). To see that $\mathbf{g_{hch}}$ has sectional curvature bounded above by a negative constant we use equation (2.9). We include this as the next Theorem.

Theorem 3.2. *The sectional curvature of* $\mathbf{g_{hch}}$ *is bounded above by a negative constant.*

Proof. Let $\sigma \subset T_p\mathbb{CH}^n$ be a 2-plane as described in equation (2.8). Then plugging the values from Lemma 3.1 into equation (2.9) yields:

$$\begin{aligned}
K(\sigma) &= (a_1^2 + a_3^2 + a_2^2b_3^2)\left(-1 + \frac{\sinh^2(r)}{\cosh^4(r)}\right) - a_4^2b_3^2 - a_4^2b_1^2 \\
&\quad + a_2^2b_1^2\left(-1 - \frac{3c_{12}^2}{4\cosh^2(r)} - \frac{3c_{12}^2\sinh^2(r)}{4\cosh^4(r)}\right) + 3a_2a_4b_1b_3\frac{c_{12}}{\cosh^3(r)} \\
&= -1 + (a_1^2 + a_3^2 + a_2^2b_3^2)\frac{\sinh^2(r)}{\cosh^4(r)} - a_2^2b_1^2\left(\frac{3c_{12}^2}{4\cosh^2(r)} + \frac{3c_{12}^2\sinh^2(r)}{4\cosh^4(r)}\right) \qquad (3.1) \\
&\quad + 3a_2a_4b_1b_3\frac{c_{12}}{\cosh^3(r)}
\end{aligned}$$

where the second equality follows from the fact that $a_1^2+a_2^2+a_3^2 = a_4^2 = 1 = b_1^2+b_3^2$.

Two remarks about (3.1). The first remark is that the first and third summands are always negative, the second is always positive, and the fourth can be either sign.

The second remark is that each component containing a hyperbolic trigonometric function approaches zero as r approaches infinity. Thus, the only possibility for nonnegative curvature is when r is small.

Now, when $r = 0$ we have:

$$K(\sigma) = -1 - \frac{3}{4}a_2^2 b_1^2 c_{12}^2 + 3a_2 a_4 b_1 b_3 c_{12}.$$

One can check that the above equation is maximized when

$$a_2 = a_4 = b_1 = b_3 = \frac{1}{\sqrt{2}},\ c_{12} = -2 \qquad \text{[recall that } (A, B) \text{ is an ON basis]}$$

and this point is unique up to the sign of the entries. At this point, we have that

$$K(\sigma) = -1 - \frac{3}{4} + \frac{3}{2} = -\frac{1}{4}. \tag{3.2}$$

Let us now return our attention to equation (3.1). The second summand is the only other term which can contribute positive curvature. The expression $(a_1^2 + a_3^2 + a_2^2 b_3^2)$ is bounded above by 1, and it is a calculus exercise to check that the expression $\sinh^2(r)/\cosh^4(r)$ is maximized when $\sinh(r) = 1$. When $\sinh(r) = 1$ we have that $\cosh^2(r) = 2$, and therefore the second summand in (3.1) is bounded above by $1/4$. Combining this fact with (3.2) proves that $\mathbf{g_{hch}}$ has nonpositive curvature.

To show that $\mathbf{g_{hch}}$ has curvature bounded above by a negative constant, just note that the fourth summand in (3.1) is maximized when $r = 0$ whereas the second summand is maximized at $r = \sinh^{-1}(1)$. Then since $K(\sigma) < 0$ on the compact interval $[0, \sinh^{-1}(1)]$, it is bounded above by a negative constant. This same constant is an upper bound for K for all $r \in [0, \infty)$. $\square$

3.2. **The weak complexified hyperbolic metric** g_{wch}**.** The *weak complexified hyperbolic metric* $\mathbf{g_{wch}}$ is defined by setting $h(r) = \cosh(r)$ and $v(r) = 2\sinh(r)$ in equation (2.5), and using the structure constants from the hyperbolic metric (that is, setting all structure constants defined in (2.3) identically equal to zero). So as a metric, we have that

$$\mathbf{g_{wch}} = \cosh^2(r)\mathbf{c_{n-1}} + \sinh^2(r)d\theta^2 + dr^2.$$

But $\mathbf{g_{wch}} \neq \mathbf{g_{hch}}$ due to the differences in the structure constants. The name of this metric comes from the fact that it has the same structure constants and coefficients as the hyperbolic metric. If one replaced $\mathbf{c_{n-2}}$ with $\mathbf{h_{2n-2}}$ in the above equation, then we would have that $\mathbf{g_{wch}} = \mathbf{h_{2n}}$.

Setting $c_i = c_k = 0$ in Theorem 2.3 proves the following Lemma.

Lemma 3.3. *Using the notation of Theorem 2.3, we have:*

$$
\begin{aligned}
&R^{wch}_{i,j,i,j} = -1 \qquad R^{wch}_{i,2n-1,i,2n-1} = -1 \qquad R^{wch}_{i,2n,i,2n} = -1\\
&R^{wch}_{i,i+1,i,i+1} = -1 - \frac{3}{\cosh^2(r)} \qquad R^{wch}_{2n-1,2n,2n-1,2n} = -1\\
&R^{wch}_{i,i+1,2n-1,2n} = R^{wch}_{i,2n-1,i+1,2n} = R^{wch}_{i,2n,i+1,2n-1} = 0\\
&R^{wch}_{i,i+1,k,k+1} = 2R^{wch}_{i,k,i+1,k+1} = -2R^{wch}_{i,k+1,i+1,k} = -\frac{2}{\cosh^2(r)}
\end{aligned}
$$

where $1 \leq i, j, k \leq 2n-2$, k is an odd integer different from i, and $j \neq i, i+1$. Also, any equations using both i and $i+1$ assume that i is an odd integer.

Notice again that every term above containing a hyperbolic trigonometric function approaches zero as r approaches infinity. Thus, $\mathbf{g_{wch}}$ is asymptotically hyperbolic.

Also, note that for all odd integers i:

$$R^{wch}_{i,i+1,i,i+1} = -1 - \frac{3}{\cosh^2(r)} < -\frac{2}{\cosh^2(r)} = R^{wch}_{i,i+1,k,k+1}.$$

So we can apply Lemma 2.4 to prove the following Corollary.

Corollary 3.4. *The sectional curvature of $\mathbf{g_{wch}}$ is bounded above by a negative constant.*

3.3. The strong complexified hyperbolic metric g_{sch}.

The *strong complexified hyperbolic metric* $\mathbf{g_{sch}}$ is defined by setting $h(r) = \cosh(r)$ and $v(r) = \sinh(2r) = 2\sinh(r)\cosh(r)$ in equation (2.5), and setting all structure constants defined in (2.3) identically equal to zero. So as a metric, we have that

$$\mathbf{g_{sch}} = \cosh^2(r)\mathbf{c_{n-1}} + \frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2.$$

The name of this metric comes from the fact that it has the same structure constants as the hyperbolic metric, but has the same coefficients as the complex-hyperbolic metric. This metric is more similar to the complex-hyperbolic metric than $\mathbf{g_{wch}}$, which explains the terminology "strong complexified" versus "weak complexified".

Setting $c_i = c_j = 0$ in Theorem 2.3, as well as inserting $h = \cosh(r)$ and $v = 2\sinh(r)\cosh(r)$, proves the following Lemma.

Lemma 3.5. *Using the notation of Theorem 2.3, we have:*

$$
\begin{aligned}
&R^{sch}_{i,j,i,j} = -1 \qquad R^{sch}_{i,2n-1,i,2n-1} = -1 - \tanh^2(r) \qquad R^{sch}_{i,2n,i,2n} = -1\\
&R^{sch}_{i,i+1,i,i+1} = -1 - \frac{3}{\cosh^2(r)} \qquad R^{sch}_{2n-1,2n,2n-1,2n} = -4\\
&R^{sch}_{i,i+1,2n-1,2n} = R^{sch}_{i,2n-1,i+1,2n} = R^{sch}_{i,2n,i+1,2n-1} = 0\\
&R^{sch}_{i,i+1,k,k+1} = 2R^{sch}_{i,k,i+1,k+1} = -2R^{sch}_{i,k+1,i+1,k} = -\frac{2}{\cosh^2(r)}
\end{aligned}
$$

where $1 \leq i,j,k \leq 2n-2$, k is an odd integer different from i, and $j \neq i, i+1$. Also, any equations using both i and $i+1$ assume that i is an odd integer.

Using the exact same reasoning as Corollary 3.4, we have the analogous Corollary.

Corollary 3.6. *The sectional curvature of $\mathbf{g_{sch}}$ is bounded above by a negative constant.*

3.4. **The d-fold weak complexified hyperbolic metric g_d.** Let $d \geq 2$ be a positive integer. The *d-fold weak complexified hyperbolic metric* $\mathbf{g_d}$ is defined as $h(r) = \cosh(r)$ and $v(r) = 2d\sinh(r)$ in equation (2.5), and setting all structure constants defined in (2.3) identically equal to zero. So as a metric, we have that

$$\mathbf{g_d} = \cosh^2(r)\mathbf{c_{n-1}} + d^2\sinh^2(r)d\theta^2 + dr^2.$$

Note that this metric has angle equal to $2d\pi$ about the core copy of $\mathbb{CH}^{n-1}$, and so is therefore only defined on $E \times (0,\infty)$. Moreover, for the metric to be complete, we need to restrict it to $E \times (r_0, \infty)$ for some $r_0 > 0$.

Setting $c_i = c_j = 0$ in Theorem 2.3, as well as inserting $h = \cosh(r)$ and $v = 2d\sinh(r)$, proves the following Lemma.

Lemma 3.7. *Using the notation of Theorem 2.3, we have:*

$$R^d_{i,j,i,j} = -1 \qquad R^d_{i,2n-1,i,2n-1} = -1 \qquad R^d_{i,2n,i,2n} = -1$$

$$R^d_{i,i+1,i,i+1} = -1 - \frac{3}{\cosh^2(r)} \qquad R^d_{2n-1,2n,2n-1,2n} = -1$$

$$R^d_{i,i+1,2n-1,2n} = R^d_{i,2n-1,i+1,2n} = R^d_{i,2n,i+1,2n-1} = 0$$

$$R^d_{i,i+1,k,k+1} = 2R^d_{i,k,i+1,k+1} = -2R^d_{i,k+1,i+1,k} = -\frac{2}{\cosh^2(r)}$$

where $1 \leq i,j,k \leq 2n-2$, k is an odd integer different from i, and $j \neq i, i+1$. Also, any equations using both i and $i+1$ assume that i is an odd integer.

Notice that the components of the curvature tensors for $\mathbf{g_d}$ and $\mathbf{g_{wch}}$ coincide. Thus, we have the following Corollary.

Corollary 3.8. *The sectional curvature of $\mathbf{g_d}$ is bounded above by a negative constant.*

4. Proof of Theorem 1.1

Let M be a Riemannian manifold modeled on $\mathbb{CH}^n$, and let N be a codimension two totally geodesic submanifold of M modeled on $\mathbb{CH}^{n-1}$. In this Section we describe a smooth Riemannian metric $\mathbf{g}$ on $E \times [0,\infty) \cong \mathbb{R}^{2n-2} \times \mathbb{S}^1 \times [0,\infty)$ whose curvature is bounded above by a negative constant, and which "descends" to the degree d ramified covering of M about N. The existence of such a metric proves Theorem 1.1.

An outline of the construction of $\mathbf{g}$ is as follows. Let $0 < r_0 < r_1 < r_2 < r_3 < r_4$. The metric $\mathbf{g}$ is constructed via the following four (really six) steps:

Step 0: Set $\mathbf{g} := \mathbf{g_{hch}}$ on the region $E \times [0, r_0]$.
Step 1: The metric $\mathbf{g}$ "unbends" from $\mathbf{g_{hch}}$ to $\mathbf{g_{wch}}$ over $E \times (r_0, r_1]$.
Step 2: The metric $\mathbf{g}$ "increases in angle" from $\mathbf{g_{wch}}$ to $\mathbf{g_d}$ over $E \times (r_1, r_2]$.

At this point, the metric $\mathbf{g}$ restricted to $\mathbb{R}^{2n-2} \times \mathbb{S}^1 \times \{r_2\}$ has total angle of $2d\pi$ about the core copy of $\mathbb{R}^{2n-2}$. Subdivide $\mathbb{S}^1$ into d equidistant arcs, and label them $A_1, \ldots, A_d$. Then for each $1 \leq l \leq d$, the metric $\mathbf{g}$ restricted to $\mathbb{R}^{2n-2} \times A_l \times \{r_2\}$ has total angle 2π about $\mathbb{R}^{2n-2}$. Moreover, $\mathbf{g_d}$ restricted to a sector A_l is identical to $\mathbf{g_{wch}}$ over all of $\mathbb{S}^1$. So in the last two steps we restrict our attention to $\mathbb{R}^{2n-2} \times A_l$, where $\mathbf{g} = \mathbf{g_{wch}}$. We then perform the same procedure for each l.

Step 3: The metric $\mathbf{g}$ "warps" from $\mathbf{g_{wch}}$ to $\mathbf{g_{sch}}$ over $\mathbb{R}^{2n-2} \times A_l \times (r_2, r_3]$.
Step 4: The metric $\mathbf{g}$ "bends" from $\mathbf{g_{sch}}$ to $\mathbf{c_n}$ over $\mathbb{R}^{2n-2} \times A_l \times (r_3, r_4]$.
Step 5: Set $\mathbf{g} = \mathbf{c_n}$ over $\mathbb{R}^{2n-2} \times A_l \times (r_4, \infty)$.

The constants r_0 through r_4 are chosen sufficiently large at each step to ensure that we can keep the curvature of $\mathbf{g}$ bounded above by a negative constant. Steps 0 and 5 are straightforward. In the next four Subsections we describe Steps 1 through 4, and verify that the curvature of $\mathbf{g}$ is bounded above by a negative constant.

4.1. **Step 1: Unbending $\mathbf{g_{hch}}$ to $\mathbf{g_{wch}}$ over $(r_0, r_1]$.** Let us first note that this Subsection is very similar to the proof of Theorem 4.1 in [LMMT].

Let P be a codimension two plane in $\mathbb{R}^{2n}$, and write $R^{2n} \cong P \times \mathbb{S}^1 \times [0, \infty) = E \times [0, \infty) =: X$. Recall from Section 2 that there exist orthogonal projections $\phi_h : \mathbb{H}^{2n} \to \mathbb{H}^{2n-2}$ and $\phi_c : \mathbb{CH}^n \to \mathbb{CH}^{n-1}$. Let $E_h(r)$ and $E_c(r)$ denote the r-tubes about $\mathbb{H}^{2n-2}$ and $\mathbb{CH}^{n-1}$, respectively. Since $P \cong \mathbb{H}^{2n-2} \cong \mathbb{CH}^{n-1}$ is contractible, we have that $E_h(r) \cong E_c(r) \cong E$ (and where " $\cong$ " means "diffeomorphic"). Using the orthogonal projections ϕ_h and ϕ_c, fix diffeomorphisms from $(\mathbb{H}^{2n}, \mathbb{H}^{2n-2}) \to (X, P)$ and $(\mathbb{CH}^n, \mathbb{CH}^{n-1}) \to (X, P)$ which are "radially isometries". By a *radial isometry* we mean the following. Let $p \in \mathbb{H}^{2n-2}$ (resp. $p \in \mathbb{CH}^{n-1}$) and let $f : \mathbb{H}^{2n} \to X$ denote the radial isometry. Then we require that $f(E_h(r) \cap \phi_h^{-1}(p)) = (f(p), \mathbb{S}^1, \{r\})$ for all such p and r, and that this map is a linearly parameterized isometry of $\mathbb{S}^1$. Note that via these identifications of $(\mathbb{H}^{2n}, \mathbb{H}^{2n-2})$ and $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$ with (X, P), it now makes sense to talk about the orthogonal projections ϕ_h and ϕ_c as functions from X to P.

Let $p \in P$ be arbitrary, and let $q = (p, \theta, r) \in X$ for some generic θ and r. Fix an orthonormal frame $(\check{X}_i^h)_{i=1}^{2n-2}$ of $\mathbb{H}^{2n-2}$ which satisfies $[\check{X}_i^h, \check{X}_j^h]_p = 0$ for all i, j. Then define the frame $(X_i^h)_{i=1}^{2n-2}$ over all of X by $X_i^h := d\phi_h^{-1}\check{X}_i^h$. Via the considerations in Subsection 2.1 we have that for all i, j:

$$0 = [\partial\theta, \partial r] = [X_i^h, \partial\theta] = [X_i^h, \partial r] = [X_i^h, X_j^h]$$

where $\partial\theta = \frac{\partial}{\partial\theta}$ and $\partial r = \frac{\partial}{\partial r}$.

Now fix a holomorphic frame $(\check{X}_i^c)_{i=1}^{2n-2}$ of $\mathbb{CH}^{n-1}$ which also satisfies that $[\check{X}_i^c, \check{X}_j^c]_p = 0$ for all i, j. Use this frame to analogously define $(X_i^c)_{i=1}^{2n-2}$ over all of X. Then from Subsection 2.2 we know that

$$0 = [\partial\theta, \partial r] = [X_i^c, \partial\theta] = [X_i^c, \partial r] = [X_i^c, X_j^c] \qquad \text{(where } |i-j| \neq 1\text{)}$$
$$2 = [X_i^c, X_{i+1}^c] \qquad \text{(where } i \text{ is an odd integer).}$$

For any $w = (q, r) \in E \times (0, \infty)$, fix the ordered basis $\beta = (X_1^h, X_2^h, \ldots, X_{2n-2}^h, \partial\theta)$ of $T_q E$. Let $\mathbf{Gr}(2n-2, \mathbb{R}^{2n-1})$ denote the Grassmannian manifold of $(2n-2)$-planes in $\mathbb{R}^{2n-1}$. Then for any $r \in (0, \infty)$ there exists a natural map $\varphi : E \times \{r\} \to \mathbf{Gr}(2n-2, \mathbb{R}^{2n-1})$ which sends the point w to the $(2n-2)$-plane spanned by $(X_1^c, \ldots, X_{2n-2}^c)_q$. Fixing the basis β gives a diffeomorphism from $\mathbf{Gr}(2n-2, \mathbb{R}^{2n-1})$ to $\mathbb{RP}^{2n-2}$, and so we can think of the map $\varphi : E \times \{r\} \to \mathbb{RP}^{2n-2}$.

The fact that $[X_i^c, \partial r] = 0$ tells us that this map φ is independent of the chosen r. Also, $[X_i^c, \partial\theta] = 0$ gives that φ is constant in the $\mathbb{S}^1$ factor of E. Thus, φ factors through P. Therefore, because P is contractible, there exists a contraction $h_r : P \to \mathbb{RP}^{2n-2}$, $r \in [r_1, r_2]$, such that $h_{r_1}(p) = \varphi(p)$ and $h_{r_2}(p) = n$ for all $p \in P$. Here, n is the point in $\mathbb{RP}^{2n-2}$ representing the line $\langle \frac{\partial}{\partial\theta} \rangle$, which corresponds to the plane $\langle X_1^h, \ldots, X_{2n-2}^h \rangle$ in $\mathbf{Gr}(2n-2, \mathbb{R}^{2n-1})$. For any $r \in [r_1, r_2]$, $p \in P$, and $q \in X$ of the form (p, θ, r), let $P_r(p)$ denote the (2n-2)-plane in $T_q X$ corresponding to $h_r(p)$.

We now define new vector fields $(X_i)_{i=1}^{2n-2}$ on X as follows. On $E \times (0, r_1]$ we define $X_i := X_i^c$. On $E \times [r_2, \infty)$ we define $X_i := X_i^h$. For $r \in (r_1, r_2)$ we vary X_i from X_i^c to X_i^h via the homotopy h_r. There are many ways to vary X_i from X_i^c to X_i^h. We require that that our choice satisfies the following conditions:

1. $X_i(q) \in P_r(p)$ for $q = (p, \theta, r)$ for any θ and i.
2. The collection (X_i) remains linearly independent throughout the process.
3. $[X_i, \partial\theta] = 0$ for all i. We can do this since $P_r(p)$ is invariant under θ.
4. $[X_i, X_j]_q$ has no X_k component for all $1 \leq k \leq 2n-2$. We can do this since $[\check{X}_i^a, \check{X}_j^a]_p = 0$ for $a = h, c$ and for all i and j.
5. For $1 \leq i < 2n-2$ an odd integer and $j \neq i+1$, $[X_i, X_j]$ has no $\frac{\partial}{\partial\theta}$ component. We can do this since $[X_i^a, X_j^a]$ has no $\frac{\partial}{\partial\theta}$ component for $a = h, c$.
6. $|[X_i, \partial r]| < \delta$ for some fixed $\delta > 0$. We can do this by choosing r_1 sufficiently large and by varying the vector fields sufficiently slowly.

Then define the metric $\mathbf{g_1}$ by

$$\mathbf{g_1} = \cosh^2(r) \left(dX_1^2 + \ldots + dX_{2n-2}^2 \right) + \sinh^2(r) d\theta^2 + dr^2 \tag{4.1}$$

where dX_i^2 denotes the covector field dual to X_i. Note that $(dX_1^2 + \ldots + dX_{2n-2}^2) = \mathbf{c_{n-1}}$, but we use the form in equation (4.1) to emphasize that the structure constants are changing as r varies from r_0 to r_1. Restricting the metric $\mathbf{g_1}$ to the interval $(r_0, r_1]$ gives the metric $\mathbf{g}$ for Step 1.

All that is left to show is that the sectional curvature of $\mathbf{g_1}$, restricted to $[r_0, r_1]$, is bounded above by a negative constant. This is the content of the next Lemma.

Lemma 4.1. *The sectional curvature of $\mathbf{g_1}$, restricted to $[r_0, r_1]$, is bounded above by a negative constant for r_0 and r_1 chosen sufficiently large.*

Proof. We need to compute the components of the $(4, 0)$ curvature tensor R_1 to $\mathbf{g_1}$. Due to conditions (3) through (5) above for the vector fields (X_i), this computation is identical to the curvature computation for Theorem 2.3 but with more restricted values for the structure constants c_i. Due to condition (6) above these computations will not be exact, but we can make these approximations arbitrarily close by choosing δ sufficiently small. So, for r_1 sufficiently large, we can approximate the components of R_1 simply by plugging in the values $h(r) = \cosh(r)$ and $v(r) = 2\sinh(r)$ into the equations in Theorem 2.3.

Using the notation of Theorem 2.3, we have:

$$R^1_{i,j,i,j} \approx -1 \qquad R^1_{i,2n-1,i,2n-1} \approx -1 + \frac{c_i^2 \sinh^2(r)}{4\cosh^4(r)}$$
$$R^1_{i,i+1,i,i+1} \approx -1 - \frac{3}{\cosh^2(r)} - \frac{3c_i^2 \sinh^2(r)}{4\cosh^4(r)}$$
$$R^1_{i,2n,i,2n} \approx -1 \qquad R^1_{2n-1,2n,2n-1,2n} \approx -1$$
$$R^1_{i,i+1,2n-1,2n} = 2R^1_{i,2n-1,i+1,2n} = -2R^1_{i,2n,i+1,2n-1} \approx \frac{-c_i}{\cosh^3(r)}$$
$$R^1_{i,i+1,k,k+1} = 2R^1_{i,k,i+1,k+1} = -2R^1_{i,k+1,i+1,k} \approx -\frac{2}{\cosh^2(r)} - \frac{c_i c_k \sinh^2(r)}{4\cosh^4(r)}$$

where $1 \leq i,j,k \leq 2n-2$, k is an odd integer different from i, and $j \neq i, i+1$. Also, recall that any equations using both i and $i+1$ assume that i is an odd integer.

Note that, for r sufficiently large, all mixed terms above approach 0 whereas the sectional curvatures of all "coordinate planes" approach -1. Thus, by Lemma 2.4, the sectional curvature of $\mathbf{g_1}$ restricted to $[r_0, r_1]$ is bounded above by a negative constant when r_0 is chosen sufficiently large.

□

4.2. **Step 2: Increasing the angle from $\mathbf{g_{wch}}$ to $\mathbf{g_d}$ over** $(r_1, r_2]$**.** This is the easiest of the four steps. Define $v(r)$ to be $2\sigma(r)$, where $\sigma(r)$ is from Lemma 2.1 in [GT87]. More specifically, $v(r)$ is a smooth positive function which satisfies the following:

(1) $v'(r) > 0$ and $v''(r) > 0$ for all $r > 0$.
(2) $v(r) = 2\sinh(r)$ for $0 \leq r \leq r_1$ and $v(r) = 2d\sinh(r)$ for $r \geq r_2$.
(3) We have $v''(r)/v(r) \approx 1$ for all r, and where this approximation can be made arbitrarily close by taking $r_2 - r_1$ arbitrarily large.
(4) We have
$$\frac{v'(r)\sinh(r)}{v(r)\cosh(r)} \approx 1 \text{ for all } r$$
and where this approximation can also be made arbitrarily close by taking $r_2 - r_1$ sufficiently large.

This choice of $v(r)$ clearly interpolates between $\mathbf{g_{wch}}$ and $\mathbf{g_d}$ by condition (2) above. Conditions (3) and (4) guarantee that all sectional curvatures stay arbitrarily close to those of $\mathbf{g_{wch}}$ (and, equivalently, $\mathbf{g_d}$). An important point to note is that, in the curvature equations in Theorem 2.3, all of the terms that have a v^2 in the numerator and an h^4 in the denominator are multiplied by c_i. Thus these terms are all zero in the metrics $\mathbf{g_{wch}}$ and $\mathbf{g_d}$, and therefore replacing $v = 2\sinh(r)$ with $v = 2d\sinh(r)$ does not change the values of the sectional curvature tensor. Then, inserting the above definition of $v(r)$ into equation (2.5) defines $\mathbf{g}$ over the region $(r_1, r_2]$.

4.3. **Step 3: Warping $\mathbf{g_{wch}}$ to $\mathbf{g_{sch}}$ over $(r_2, r_3]$.** Arbitrarily C^2-small changes to the coefficients of a metric produce arbitrarily small changes to the sectional curvature tensor. So, for r_2 chosen sufficiently large, we can make the approximations

$$\cosh(r) \approx \frac{1}{2}e^r \approx \sinh(r).$$

in order to simplify all curvature computations. Recall that

$$\mathbf{g_{wch}} = \cosh^2(r)\mathbf{c_{n-1}} + \sinh^2(r)d\theta^2 + dr^2 \approx \frac{1}{4}e^{2r}\mathbf{c_{n-1}} + \frac{1}{4}e^{2r}d\theta^2 + dr^2$$
$$\mathbf{g_{sch}} = \cosh^2(r)\mathbf{c_{n-1}} + \frac{1}{4}\sinh^2(2r)d\theta^2 + dr^2 \approx \frac{1}{4}e^{2r}\mathbf{c_{n-1}} + \frac{1}{4}\left(\frac{1}{4}e^{4r}\right)d\theta^2 + dr^2.$$

Fix small positive constants $\varepsilon_f, \varepsilon_g > 0$ whose size will depend on the size of r_2. Let $f(r)$ and $g(r)$ be positive, smooth, real-valued functions defined on $[0, \infty)$ which satisfy the following:

(1) $f(r) = 1$ for $r \leq r_2$, $f(r) = \frac{1}{2}$ for $r \geq r_3$, and f is non-increasing.
(2) $|f'(r)|, |f''(r)| < \varepsilon_f$ for all r.
(3) $g(r) = r$ for $r \leq r_2$, $g(r) = 2r$ for $r \geq r_3$, and g is non-decreasing.
(4) $1 \leq g'(r) \leq 2 + \varepsilon_g$ and $|g''(r)| < \varepsilon_g$ for all r.

It is clear that, for r_3 chosen sufficiently large, such functions f and g exist for arbitrarily small (but fixed) ε_f and ε_g.

Now define

$$v_3(r) = f(r)\left(e^{g(r)} - e^{-g(r)}\right) \approx f(r)e^{g(r)}$$

and

$$\mathbf{g_3} = \cosh^2(r)\mathbf{c_{n-1}} + \frac{1}{4}v_3^2(r)d\theta^2 + dr^2 \approx \frac{1}{4}e^{2r}\mathbf{c_{n-1}} + \frac{1}{4}\left(f(r)e^{g(r)}\right)^2 d\theta^2 + dr^2.$$

By construction, the metric $\mathbf{g_3}$ interpolates between $\mathbf{g_{wch}}$ and $\mathbf{g_{sch}}$ over the interval $(r_2, r_3]$. So the metric $\mathbf{g}$ in Step 3 will just be $\mathbf{g_3}$ restricted to this interval. All that is left to show is that the sectional curvature of $\mathbf{g_3}$ is bounded above by a negative constant.

Lemma 4.2. *The sectional curvature of $\mathbf{g_3}$, restricted to $[r_2, r_3]$, is bounded above by a negative constant for r_2 and r_3 chosen sufficiently large.*

Proof. Using the notation of Theorem 2.3, and recalling that all structure constants c_i are identically zero in both metrics $\mathbf{g_{wch}}$ and $\mathbf{g_{sch}}$, the components of the $(4,0)$ curvature tensor R^3 for $\mathbf{g_3}$ are:

$$R^3_{i,j,i,j} \approx -1 - \frac{4}{e^{2r}} \qquad R^3_{i,2n-1,i,2n-1} \approx -\frac{f'}{f} - g'$$
$$R^3_{i,i+1,i,i+1} \approx -1 - \frac{16}{e^{2r}} \qquad R^3_{i,2n,i,2n} = -1$$
$$R^3_{2n-1,2n,2n-1,2n} \approx -\frac{1}{f}\left(f'' + 2f'g'\right) - (g')^2 - g''$$
$$R^3_{i,i+1,2n-1,2n} = 2R^3_{i,2n-1,i+1,2n} = -2R^3_{i,2n,i+1,2n-1} = 0$$
$$R^3_{i,i+1,k,k+1} = 2R^3_{i,k,i+1,k+1} = -2R^3_{i,k+1,i+1,k} \approx -\frac{8}{e^{2r}}.$$

Fix $\frac{1}{4} > \delta > 0$. Then we may choose r_2 sufficiently large and $\varepsilon_f, \varepsilon_g$ sufficiently small (by choosing r_3 sufficiently large), so that

$$\begin{aligned}
&-\delta < -|R^3_{i,i+1,k,k+1}| < 0 \\
&-4-\delta < R^3_{2n-1,2n,2n-1,2n} < -1+\delta \\
&-2-\delta < R^3_{i,2n-1,i,2n-1} < -1+\delta.
\end{aligned}$$

Lemma 2.4 then proves that the sectional curvature of $\mathbf{g_3}$ is bounded above by a negative constant. □

4.4. **Step 4: Bending $\mathbf{g_{sch}}$ to $\mathbf{c_n}$ over** $(r_3, r_4]$**.** Recall that both $\mathbf{g_{sch}}$ and $\mathbf{c_n}$ are equal to

$$\cosh^2(r)\mathbf{c_{n-1}} + \frac{1}{4}\sinh^2(r)d\theta^2 + dr^2 \approx \frac{1}{4}e^{2r}\mathbf{c_{n-1}} + \frac{1}{4}\left(\frac{1}{4}e^{4r}\right)d\theta^2 + dr^2. \tag{4.2}$$

The only difference in the metrics is the value of the structure constants. Using the notation of Subsection 2.3, each structure constant c_i has value 0 in $\mathbf{g_{sch}}$ whereas it has value 2 with respect to $\mathbf{c_n}$.

So we need to "bend" the vector fields (X_i) so that, for i an odd integer, the Lie bracket $[X_i, X_{i+1}]$ varies from 0 to $2\frac{\partial}{\partial\theta}$. We do this exactly as in Subsection 4.1, but in reverse. So the metric $\mathbf{g_4}$ is defined as in (4.2), but with Lie brackets defined via (1) through (6) from Subsection 4.1 (in reverse). So all that is left to show is that the sectional curvature of $\mathbf{g_4}$, restricted to $[r_3, r_4]$, is bounded above by a negative constant.

Lemma 4.3. *The sectional curvature of* $\mathbf{g_4}$*, restricted to* $[r_3, r_4]$*, is bounded above by a negative constant for* r_3 *and* r_4 *chosen sufficiently large.*

Proof. As explained in the proof of Lemma 4.1, we can approximate the components of the $(4,0)$ curvature tensor R_4 to $\mathbf{g_4}$ using the equations in Theorem 2.3 when choosing r_4 sufficiently large. Then, for r_3 sufficiently large, we can approximate the components of R_4 by plugging in the values $h(r) \approx \frac{1}{2}e^r$ and $v(r) \approx \frac{1}{2}e^{2r}$.

Using the notation of Theorem 2.3, we have:

$$\begin{aligned}
&R^4_{i,j,i,j} \approx -1 - \frac{4}{e^{2r}} && R^4_{i,2n-1,i,2n-1} \approx -2 + \frac{c_i^2}{4} \\
&R^4_{i,i+1,i,i+1} \approx -1 - \frac{16}{e^{2r}} - \frac{3c_i^2}{4} && \\
&R^4_{i,2n,i,2n} \approx -1 && R^4_{2n-1,2n,2n-1,2n} \approx -4 \\
&R^4_{i,i+1,2n-1,2n} = 2R^4_{i,2n-1,i+1,2n} = -2R^4_{i,2n,i+1,2n-1} \approx -c_i && \\
&R^4_{i,i+1,k,k+1} = 2R^4_{i,k,i+1,k+1} = -2R^4_{i,k+1,i+1,k} \approx -\frac{8}{e^{2r}} - \frac{c_i c_k}{2} &&
\end{aligned}$$

where $1 \le i, j, k \le 2n-2$, k is an odd integer different from i, and $j \ne i, i+1$. Also, recall that any equations using both i and $i+1$ assume that i is an odd integer.

We will apply Lemma 2.4 to finish the proof. Of the inequalities necessary to apply Lemma 2.4, all but three are immediately obvious. The three that are not obvious at first are:

(1) $R^1_{i,i+1,i,i+1} \le -|R^1_{i,i+1,k,k+1}|$

(2) $R^1_{i,i+1,i,i+1} \leq -|R^1_{i,i+1,2n-1,2n}|$
(3) $R^1_{i,2n-1,i,2n-1} \leq -|R^1_{i,2n-1,i+1,2n}|$

We deal with each individually. In what follows, we assume that all structure constants have values in $[0,2]$ in order to remove the absolute value signs.

For inequality (1), we have

$$\begin{aligned} R^1_{i,i+1,i,i+1} \leq -|R^1_{i,i+1,k,k+1}| \qquad &\iff \qquad -1 - \frac{16}{e^{2r}} - \frac{3c_i^2}{4} \leq \frac{-8}{e^{2r}} - \frac{c_i c_k}{2} \\ &\iff \qquad -1 - \frac{8}{e^{2r}} - \frac{c_i}{4}\left(3c_i - 2c_k\right) \leq 0. \end{aligned}$$

Now, the worst case scenario is when $c_k = 2$. Also, for r large, $8/e^{2r} \approx 0$. So setting $c_k = 2$ and $8/e^{2r} = 0$, we need to show that

$$-1 + c_i - \frac{3}{4}c_i^2 \leq 0 \qquad \iff \qquad 0 \leq 3c_i^2 - 4c_i + 4.$$

From here, a simple calculus argument shows that the polynomial $3c_i^2 - 4c_i + 4$ is always positive.

For inequality (2), we have

$$R^1_{i,i+1,i,i+1} \leq -|R^1_{i,i+1,2n-1,2n}| \qquad \iff \qquad -1 - \frac{16}{e^{2r}} - \frac{3c_i^2}{4} \leq -c_i.$$

As above, we set $16/e^{2r} = 0$. The resulting inequality is then equivalent to

$$0 \leq 3c_i^2 - 4c_i + 4$$

which is the same polynomial that appeared above.

Finally, for inequality (3) we need to show

$$\begin{aligned} R^1_{i,2n-1,i,2n-1} \leq -|R^1_{i,2n-1,i+1,2n}| \qquad &\iff \qquad -2 + \frac{c_i^2}{4} \leq -\frac{c_i}{2} \\ &\iff \qquad c_i^2 + 2c_i - 8 \leq 0. \end{aligned}$$

Now, the polynomial $c_i^2 + 2c_i - 8$ is increasing on $[0,2]$, and so has a maximum value of 0 at $c_i = 2$. Thus, inequality (3) always holds, proving the Lemma.

$\square$

5. Preliminaries

5.1. **Formula for the curvature tensor of $\mathbb{CH}^n$ in terms of the complex structure J.** The components of the (4,0) curvature tensor of the complex hyperbolic metric g can be expressed in terms of g and the complex structure J. The following formula can be found in [KN96] or in Section 5 of [Bel11]. In this formula X, Y, Z, and W are arbitrary vector fields.

$$\begin{aligned} \langle R_g(X,Y)Z, W\rangle_g = {} & \langle X, W\rangle_g \langle Y, Z\rangle_g - \langle X, Z\rangle_g \langle Y, W\rangle_g \\ & + \langle X, JW\rangle_g \langle Y, JZ\rangle_g - \langle X, JZ\rangle_g \langle Y, JW\rangle_g + 2\langle X, JY\rangle_g \langle W, JZ\rangle_g. \end{aligned} \tag{5.1}$$

5.2. **General curvature formulas for warped product metrics.** The curvature formulas below, which were worked out by Belegradek in [Bel12] and stated in Appendix B of [Bel11], apply to metrics of the form $g = g_r + dr^2$ on manifolds of the form $E \times I$ where I is an open interval and E is a manifold. The formulas are true provided that for each point $q \in E$ there exists a local frame $\{X_i\}$ on a neighborhood U_q in E which is g_r-orthogonal for each r. Such a family of metrics (E, g_r) is called *simultaneously diagonalizable*. Let

$$h_i(r) := \sqrt{g_r(X_i, X_i)}.$$

Then the local frame $\{Y_i\}$ defined by

$$Y_i = \frac{1}{h_i} X_i$$

is a g_r-orthonormal frame on U_q for any value of r. We then have the following formulas for the (4,0) curvature tensor R_g in terms of the (4,0) curvature tensor R_{g_r}, the collection $\{h_i\}$, and the Lie brackets $[Y_i, Y_j]$. Note that $\langle , \rangle$ is used to denote the metric g and $\partial r = \frac{\partial}{\partial r}$.

$$\langle R_g(Y_i, Y_j)Y_i, Y_j\rangle = \langle R_{g_r}(Y_i, Y_j)Y_i, Y_j\rangle - \frac{h_i' h_j'}{h_i h_j} \tag{5.2}$$

$$\langle R_g(Y_i, Y_j)Y_k, Y_l\rangle = \langle R_{g_r}(Y_i, Y_j)Y_k, Y_l\rangle \quad \text{if } \{i, j\} \neq \{k, l\} \tag{5.3}$$

$$\langle R_g(Y_i, \partial r)Y_i, \partial r\rangle = -\frac{h_i''}{h_i} \qquad \langle R_g(Y_i, \partial r)Y_j, \partial r\rangle = 0 \quad \text{if } i \neq j \tag{5.4}$$

$$\begin{aligned} 2\langle R\left(\partial r, Y_i\right) Y_j, Y_k\rangle = {} & \langle [Y_i, Y_k], Y_j\rangle \left(\ln \frac{h_j}{h_k}\right)' + \langle [Y_j, Y_i], Y_k\rangle \left(\ln \frac{h_k}{h_j}\right)' \\ & + \langle [Y_j, Y_k], Y_i\rangle \left(\ln \frac{h_i^2}{h_j h_k}\right)'. \end{aligned} \tag{5.5}$$

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References

[Bel11] I. Belegradek, *Complex hyperbolic hyperplane complements*, Math. Ann. **353** (2012), no. 2, 545-579.

[Bel12] I. Belegradek, *Rigidity and relative hyperbolicity of real hyperbolic hyperplane complements*, Pure Appl. Math. Q. **8** (2012), no. 1, 15-51.

[Bes87] A. L. Besse, *Einstein manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **10**, Springer-Verlag, Berlin (1987).

[FJ94] F.T. Farrell and L.E. Jones, *Complex hyperbolic manifolds and exotic smooth structures*, Invent. math. **117** (1994), no. 1, 57-74.

[GT87] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), 1-12.

[KN96] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. II, Wiley Classics Library, Wiley, New York (1996). Reprint of the 1969 original.

[LMMT] J.F. Lafont, J. Meyer, B. Minemyer, and B. Tshishiku, *New examples of negatively curved 4-manifolds*, preprint.

[Min16] B. Minemyer, *Real hyperbolic hyperplane complements in the complex hyperbolic plane*, preprint, arxiv: 1609.01974.

[Min17a] B. Minemyer, *Negatively curved codimension one distributions with spherical ends*, preprint.

[Min17b] B. Minemyer, *Warped product metrics on (complex) hyperbolic manifolds*, preprint.

[Ont15] P. Ontaneda, *On the Farrell and Jones warping deformation*, J. Lond. Math. Soc. **92** (2015), no. 3, 566-582.

[Zhe96] F. Zheng, *Examples of non-positively curved Kähler manifolds*, Comm. Anal. Geom. **4** (1996), no. 1-2, 129-160.

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